Plan

1 Integration

① Integration

Defn. An antiderivative $F(x)$ of a function $f(x)$ is a function such that $F'(x) = f(x)$.

Ex: $f(x) = 2x$ fn. $F(x) = x^2$ or $F(x) = x^2 + 2$ antiderivative

The general antiderivative of $f(x)$ is $F(x) + C$ if $F'(x) = f(x)$. C = undetermined constant

Defn. Indefinite integral $\int f(x)\,dx$ = general antiderivative of $f(x)$

integration symbol

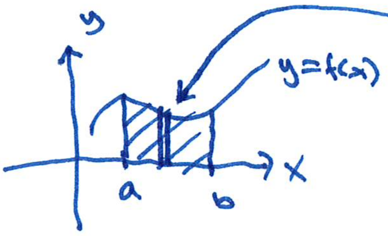
x is the integration variable

Ex: $\int 2x\,dx = \underline{\underline{x^2 + C}}$ integration constant

Used in differential equations.

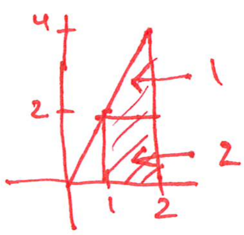
Definite integral $\int_a^b f(x)\,dx = \left[F(x) + C\right]_a^b$

area



$= (F(b) + \cancel{C}) - (F(a) + \cancel{C})$

$= F(b) - F(a)$



Ex: $\int_1^2 2x\,dx = \left[x^2\right]_1^2 = 2^2 - 1^2 = \underline{\underline{3}}$

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Computing definite integrals

Ex: $\int x^3 - 3x + 2 \, dx = \underline{\frac{1}{4}\cdot x^4 - \frac{3}{2}\cdot x^2 + 2x + C}$

$\frac{1}{\cancel{4}}\cdot \cancel{4}x^3 \quad \frac{3}{\cancel{2}}\cdot \cancel{2}x \quad 2 \quad 0$

Integration rules:

i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C = \frac{1}{n+1}\cdot x^{n+1} + C \qquad (n \neq -1)$

$\int \frac{1}{x} \, dx = \ln|x| + C$

ii) $\int u \pm v \, dx = \int u \, dx \pm \int v \, dx$

$\int c \cdot u \, dx = c \cdot \int u \, dx$

u, v expr.
c const.

iii) $\int e^x \, dx = e^x + C$

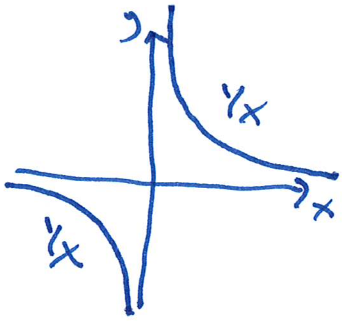
$\int a^x \, dx = a^x \cdot \frac{1}{\ln(a)} + C$

Ex: $\int 1 + x + x^2 \, dx = \underline{x + \frac{x^2}{2} + \frac{x^3}{3} + C}$

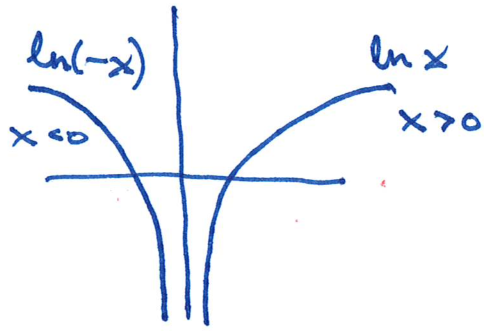
$\int \frac{x^2 - 1}{x} \, dx = \int x - \frac{1}{x} \, dx = \underline{\frac{x^2}{2} - \ln|x| + C}$

$\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C = \underline{\frac{2}{3} x\sqrt{x} + C}$

<u>Comment:</u> We know that $f(x) = \ln x$ is defined for $x > 0$ and that $f'(x) = 1/x$.



$f(x) = 1/x \;,\; x \neq 0$



$F(x)$

$$(\ln(-x))' = \frac{1}{-x} \cdot (-1) = 1/x \qquad (\ln x)' = \frac{1}{x}$$

$$F(x) = \begin{cases} \ln x \,, & x > 0 \\ \ln(-x) \,, & x < 0 \end{cases} = \ln|x|$$

<u>Ex:</u> $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = \underline{\underline{-\frac{1}{x} + C}}$

<u>Ex:</u> $\int \ln x \, dx = ?$ $\qquad \int x e^x \, dx = ?$

$\int \frac{1}{1+e^x}\,dx = ?$ $\qquad \int \frac{1}{1-x^2}\,dx = ?$

(a) Integration by parts

$$\boxed{\int u'v\,dx = uv - \int uv'\,dx}$$

$(uv)' = u'v + uv'$

$\int (uv)'dx = \int u'v + uv'\,dx$

$$\boxed{uv = \int u'v\,dx + \int uv'\,dx}$$

Ex: $\cancel{\int x \cdot e^x\,dx = \frac{x^2}{2} \cdot e^x - \int \frac{x^2}{2} \cdot e^x}$

$\cancel{\boxed{\begin{array}{ll} u = x^2/2 & v = e^x \\ u' = x & v' = e^x \end{array}}}$

$\int x \cdot e^x\,dx = x e^x - \int e^x \cdot 1\,dx = x e^x - \int e^x\,dx$

$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$

$= \underline{\underline{x e^x - e^x + C}}$

Ex: $\int x \cdot \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$

$\boxed{\begin{array}{ll} u = x^2/2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} \cdot x\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(x^2/2\right) + C$

Note: $\boxed{\int f(x) \cdot g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx}$

$= \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$

Ex: $\int \ln x\,dx = \int 1 \cdot \ln(x)\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx$

$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$

$= x \ln x - \int 1\,dx$

$= \underline{\underline{x \ln x - x + C}}$

$(x\ln x - x)' =$
$1 \cdot \ln x + x \cdot \frac{1}{x} - 1$
$= \ln x + 1 - 1 = \ln x$

(b) Substitution

Ex: $\int e^{1-2x}\,dx = \dfrac{e^{1-2x}}{-2} + C = \underline{-\frac{1}{2}e^{1-2x} + C}$

$f(x) = e^{1-2x} = e^{u}$, $u = 1-2x$

$f'(x) = e^{u}\cdot(-2) = \underline{-2\cdot e^{1-2x}}$

$$\int e^{1-2x}\,dx = \int e^{u}\,dx = \int e^{u}\frac{1}{(-2)}\,du$$

$\boxed{\begin{array}{l} u = 1-2x \\ du = -2\,dx \end{array}}$

$dx = \frac{1}{-2}\cdot du$

$$= -\frac{1}{2}\int e^{u}\,du$$

$$= -\frac{1}{2}e^{u} + C$$

$$= \underline{\underline{-\frac{1}{2}e^{1-2x} + C}}$$

$\boxed{\begin{array}{l} \underline{\text{In general:}} \\ du = u'\cdot dx \end{array}}$

$dx = \frac{1}{u'}\,du$

Ex: $\displaystyle\int \frac{x}{1-x^2}\,dx = \int \frac{\cancel{x}}{u}\cdot\frac{1}{(-2\cancel{x})}\,du = -\frac{1}{2}\int\frac{1}{u}\,du$

$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\cdot dx \end{array}}$

$dx = \frac{1}{-2x}\,du$

$$= -\frac{1}{2}\ln|u| + C$$

$$= \underline{\underline{-\frac{1}{2}\ln|1-x^2| + C}}$$

Ex: $\displaystyle\int x\cdot\sqrt{1+x^2}\,dx = \int \cancel{x}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du = \frac{1}{2}\int u^{1/2}\,du$

$\boxed{\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \end{array}}$

$$= \frac{1}{\cancel{2}}\cdot\frac{u^{3/2}}{3/\cancel{2}} + C = \underline{\frac{1}{3}(1+x^2)^{3/2} + C}$$

$$= \underline{\frac{1}{3}(1+x^2)\sqrt{1+x^2} + C}$$

(c) Integration of rational functions

Rational fn.

$$f(x) = \frac{p(x)}{q(x)} \quad \text{polyn.}$$

Ex: $\int \frac{3}{1-2x}\,dx = \int \frac{3}{u}\frac{du}{-2}$

$\boxed{\begin{array}{l} u = 1-2x \\ du = -2\,dx \end{array}}$ $= \frac{3}{-2}\int \frac{1}{u}\,du = -\frac{3}{2}\ln|1-2x| + C$

$$\int \frac{x^2}{1-x}\,dx = \int -x-1+\frac{1}{1-x}\,dx$$

$$= -\frac{x^2}{2} - x - \ln|1-x| + C$$

Polynomial division

$$x^2 : -x+1 = -x-1$$

$$\begin{array}{l} x^2 - x \\ \hline x \\ x-1 \\ \hline 1 \end{array}$$

$$\frac{x^2}{1-x} = -x-1+\frac{1}{1-x}$$

Ex: $\int \frac{2x+4}{1-x^2}\,dx$

$1-x^2 = (1-x)(1+x)$

$= \int \frac{A}{1-x}\,dx + \int \frac{B}{1+x}\,dx$

$(1-x)(1+x) \cdot \Big| \quad \frac{2x+4}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$

$2x+4 = A(1+x) + B(1-x)$

$= \int \frac{3}{1-x}\,dx + \int \frac{1}{1+x}\,dx$

$x=-1$: $2 = B \cdot 2$ $\quad B=1$

$x=1$: $6 = 2A$ $\quad A=3$

$= \frac{3}{-1}\ln|1-x| + \frac{1}{1}\ln|1+x| + C$

$\Downarrow$

$= \ln|1+x| - 3\ln|1-x| + C$

$$\frac{2x+4}{1-x^2} = \frac{3}{1-x} + \frac{1}{1+x}$$

Integration of fractions:

- substitution
- polynomial division
- partial fraction decomposition

Ex: $\int \frac{x}{(1-x)^2}\,dx$

$$= \int \frac{-1}{1-x}\,dx + \int \frac{1}{(1-x)^2}\,dx$$

$$= \frac{-1}{-1}\ln|1-x| + \frac{1}{1-x} + C$$

$$= \ln|1-x| + \frac{1}{1-x} + C$$

$$\frac{x}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2}$$

$$x = A(1-x) + B$$

$x=1$: $1 = B$ $\quad$ $B=1$

$x=0$: $0 = A\cdot 1 + B$

$A = -B = -1$

$$\int \frac{1}{(1-x)^2}\,dx = \int \frac{1}{u^2}\,\frac{du}{-1} \qquad \boxed{\begin{aligned} u &= 1-x \\ du &= -1\cdot dx \end{aligned}}$$

$$= -\int u^{-2}\,du$$

$$= -\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{1}{u} + C = \frac{1}{1-x} + C$$