Plan

1 Vectors and matrices
2 Matrix multiplication
3 Inverse matrices

① Vectors and matrices

Defn: An $m \times n$ ~~matrix~~ matrix is a rectangular array of numbers (m rows, n col's)

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix}$$

2×2 - matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

2×3

a_{ij}: entry in row i, col j of A

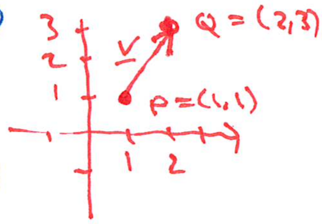
Defn An n-vector $\underline{v} = \vec{v}$ is an $n \times 1$-matrix (column vector)

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ 3-vector

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$ n-vector

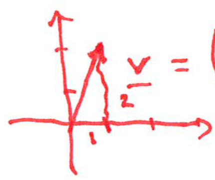
Geometric representation:

Ex: $P = (1,1)$ $Q = (2,3)$



$$\vec{PQ} = \begin{pmatrix} 2-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Ex:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} : \quad \|\underline{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \simeq 2.23$$

Length:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} : \quad \|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

length of $\underline{v}$

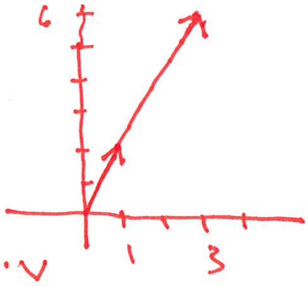
Operations: Matrices / vectors

Addition/subtraction: $A \pm B$

Ex.
$$\begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 10 & 5 \end{pmatrix}$$

$\underline{v} \pm \underline{w}$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



Scalar multiplication: $c \cdot A$

Ex:
$$3 \cdot \begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 21 & 12 \end{pmatrix}$$

$c \cdot \underline{v}$

$$3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Inner products (dot product)

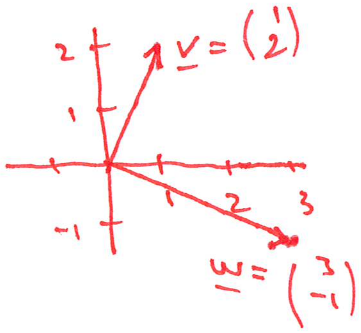
$\underline{v}, \underline{w}$: n-vectors

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 \cdot w_1 + v_2 \cdot w_2 + \cdots + v_n \cdot w_n$$

Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 1 \cdot 2 + 2 \cdot (-1) + 1 \cdot 0 = 2 - 2 + 0 = \underline{0}$$

Defn: Two n-vectors $\underline{v}$ and $\underline{w}$ are called orthogonal (perpendicular) if the angle between the vectors is 90°. We write $\underline{v} \perp \underline{w}$.



$\underline{v} \cdot \underline{w} = 1$

Result:

$$\boxed{\underline{v} \perp \underline{w} \iff \underline{v} \cdot \underline{w} = 0}$$

angle less than $90^\circ \iff \underline{v} \cdot \underline{w} > 0$

" more – " – $\iff \underline{v} \cdot \underline{w} < 0$

② Matrix multiplication $A \cdot B$

Ex:

$$\begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 \\ 7\cdot 1 + 4 \cdot 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 5 \\ 15 \end{pmatrix}}}$$

2×2 2×1

=

2×1

Result.

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}} = \begin{pmatrix} 1\cdot 1 + 2\cdot 3 & 1\cdot 0 + 2\cdot(-1) \\ 7\cdot 1 + 4\cdot 3 & 7\cdot 0 + 4\cdot(-1) \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 7 & -2 \\ 19 & -4 \end{pmatrix}}}$$

> The matrix product $A\cdot B$, where A is an $m\times n$ matrix and B is an $r\times s$ matrix, is defined when $n=r$. It gives an $m\times s$ matrix by combining the rows of A with the columns of B using "inner product".

Remarks:

- matrix multiplication is <u>not</u> position by position
- — " — is <u>not</u> symmetric; that is,

  $AB \neq BA$ in general

Ex: $\underset{2\times 1}{\begin{pmatrix}1\\2\end{pmatrix}}\cdot\underset{2\times 2}{\begin{pmatrix}1&2\\7&4\end{pmatrix}}$ is not defined

$$B\cdot A=\underset{2\times 2}{\begin{pmatrix}1&0\\3&-1\end{pmatrix}}\cdot\underset{2\times 2}{\begin{pmatrix}1&2\\7&4\end{pmatrix}}=\begin{pmatrix}1&2\\-4&2\end{pmatrix}$$

$$\neq$$

$$A\cdot B\ \begin{pmatrix}1&2\\7&4\end{pmatrix}\cdot\begin{pmatrix}1&0\\3&-1\end{pmatrix}=\begin{pmatrix}7&-2\\19&-4\end{pmatrix}$$

Ex: $\underline{v}=\begin{pmatrix}1\\2\end{pmatrix}$, $\underline{w}=\begin{pmatrix}3\\-4\end{pmatrix}$

$$\boxed{\underline{v}\cdot\underline{w}=1\cdot 3+2\cdot(-4)=-5}$$

$$\cancel{\underset{1\times 2}{\begin{pmatrix}1&2\end{pmatrix}}\cdot\underset{2\times 1}{\begin{pmatrix}3\\-4\end{pmatrix}}=\begin{pmatrix}1\cdot 3+2\cdot(-4)\end{pmatrix}=\begin{pmatrix}-5\end{pmatrix}}$$

Powers: $A^2, A^3, \dots$

$$A^2=\underset{m\times n}{A}\cdot\underset{m\times n}{A}$$

$m=n$

$n\times n$ $n\times n$

If A is an $m\times n$-matrix, we say that it is square if $m=n$.
The powers $A^2, A^3, A^4, \dots$ are defined for square matrices.

Ex: $\begin{pmatrix}1&0\\1&1\end{pmatrix}^2=\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix}=\underline{\begin{pmatrix}1&0\\2&1\end{pmatrix}}$

$\begin{pmatrix}1&0\\1&1\end{pmatrix}^3=\begin{pmatrix}1&0\\1&1\end{pmatrix}^2\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\2&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix}=\underline{\begin{pmatrix}1&0\\3&1\end{pmatrix}}$

Identity matrix I

Defn: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Result:

For any matrix A, we have: $A \cdot I = A$
$I \cdot A = A$

Ex: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}$

$$A \cdot I = \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}} \cdot \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} = A$$

$$I \cdot A = \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} = A$$

Multiplication of diagonal matrices

$$\underset{4\times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}} \cdot \underset{4\times 4}{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix}} = \underline{\underline{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 42 \end{pmatrix}}}$$

Defn: A square matrix is called diagonal if all entries not on the main diagonal are zero.

Ex:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

③ Inverse matrix : A^{-1}

Defn The inverse matrix of a matrix A is another matrix, written A^{-1}, with the following properties:

$$A^{-1} \cdot A = I$$
$$A \cdot A^{-1} = I$$

Fact: Only square matrices can have inverses, and not all square matrices have inverses.

A is called invertible if it has an inverse A^{-1}.

Ex: $A = (2)$ $A^{-1} = (1/2)$

$$A \cdot A^{-1} = (2)\cdot(1/2) = (1) = I$$
$$A^{-1} \cdot A = (1/2)\cdot(2) = (1) = I$$

$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$A^{-1} \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Method: Reduce $(A|I)$ to reduced echelon form using elem. row-op.

$$(A|I) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \\ -3 \end{matrix} \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{array}\right) \cdot (-1/7) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right) \begin{matrix} \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 1/7 & 2/7 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right)$$

Defn: A matrix is in reduced echelon form if it is in echelon form, and i) all pivots $= 1$ and ii) all entries over a pivot $= 0$.

If $(A|I) \to (B|C)$ with $B = I$, then $A^{-1} = C$
— " — with $B \neq I$, then A is not invertible.

Conclusion in ex: $A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1} = \underline{\underline{\begin{pmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{pmatrix}}}$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \frac{1}{7} \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$

$$= \frac{1}{7} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}} = \frac{1}{7} \cdot \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Connection with linear systems:

Ex: $\begin{aligned} x + 2y &= 13 \\ 3x - y &= 7 \end{aligned}$ $\xrightarrow{\text{Gaussian elim.}}$ $\left(\begin{array}{cc|c} 1 & 2 & 13 \\ 3 & -1 & 7 \end{array}\right) \to \ldots$

augmented coeff. matrix

$\downarrow$

$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 13 \\ 7 \end{pmatrix}$

coeff. matrix

$$A \cdot \underline{x} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x - y \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix} = \underline{b}$$

$\boxed{A\underline{x} = \underline{b}}$ matrix form of the linear system

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix} \quad | \; A^{-1} \cdot$$

$$\frac{1}{7}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 13 \\ 7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 27 \\ 32 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 27/7 \\ 32/7 \end{pmatrix}}}$$

$$A^{-1} \cdot A \cdot \underline{x} = A^{-1} \cdot \underline{b} \Rightarrow I \cdot \underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1} \cdot \underline{b}$$

Result: A $m \times n$ matrix

i) A is invertible $\Leftrightarrow$ A is square and $\det(A) \neq 0$

ii) If A is invertible, then A^{-1} is unique and it is given

$$A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A)$$