	Plan
1	Functions and derivatives
2	Exponential functions and logarithms
3	Higher derivatives and convexity

① Functions and derivatives — one variable case

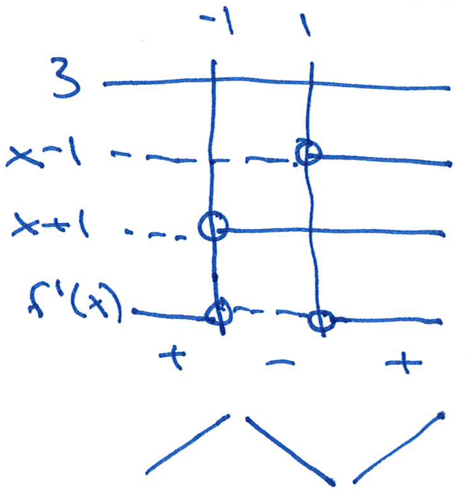
Ex:

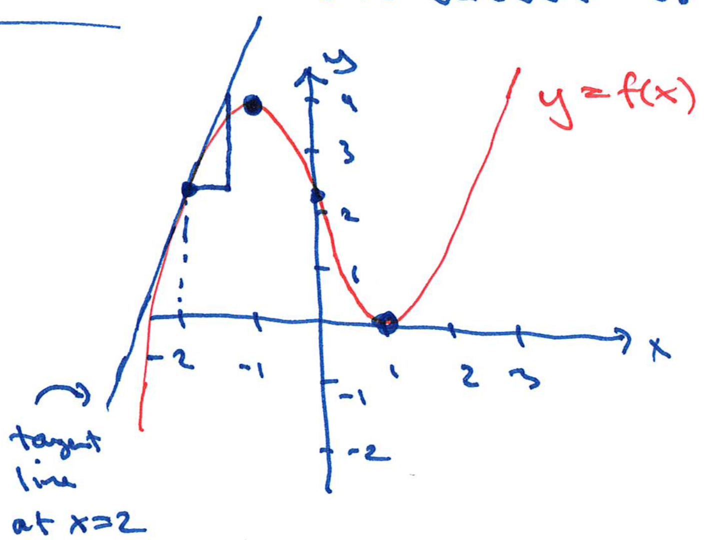
$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3$$

$$= 3(x^2 - 1)$$

$$= 3(x-1)(x+1)$$





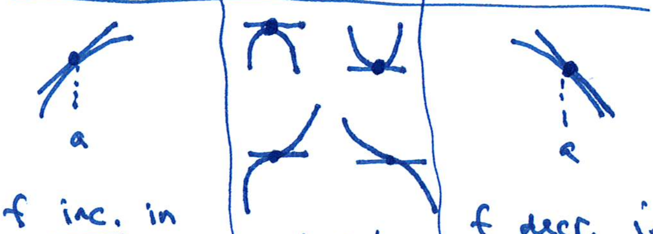
$f'(2)$ = slope of the tangent line at $x=2$

$x=-1$: local max, $f(-1) = 4$

$x=1$: local min, $f(1) = 0$

In general: at $x = a$

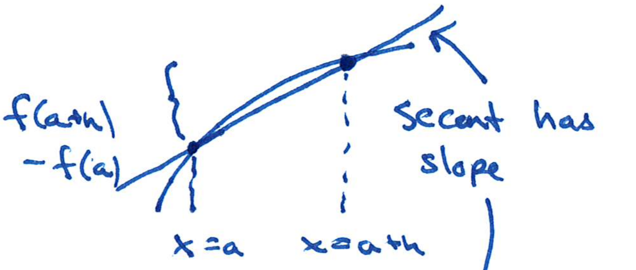
$f'(a) > 0$	$f'(a) = 0$	$f'(a) < 0$



f inc. in $(a-\varepsilon, a+\varepsilon)$	Stationary pt.	f decr. in $(a-\varepsilon, a+\varepsilon)$

Eivind Eriksen, Office B3-085y, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/FORK1003/
BI Norwegian Business School
MSc in Business/Finance

Computing derivatives:



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h}$$

Defn: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Derivation rules:

i) $f(x) = x^n \Rightarrow f'(x) = n \cdot x^{n-1}$ (holds for any const. n)

$(x^n)' = nx^{n-1}$

ii) $(u \pm v)' = u' \pm v'$ (holds for any expr. u and v, and any const. c)

$(c \cdot u)' = c \cdot u'$

iii) $(u \cdot v)' = u' \cdot v + u \cdot v'$ — " —

iv) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

v) $(a^x)' = a^x \cdot \ln(a)$ $(a>0)$

$(\log_a x)' = \frac{1}{x} \cdot \frac{1}{\ln(a)}$ $(a>0)$

vi) Chain rule: $f(x) = h(u(x)) = h(u)$ where $u = u(x)$

$f'(x) = h'(u(x)) \cdot u'(x)$

Ex: $(x^3 - 3x + 2)' = (x^3)' - (3x)' + (2)' = 3x^2 - 3(x)' + 2 \cdot (1)'$

$= 3x^2 - 3 + 0 = \underline{\underline{3x^2 - 3}}$

$(ax+b)' = a$

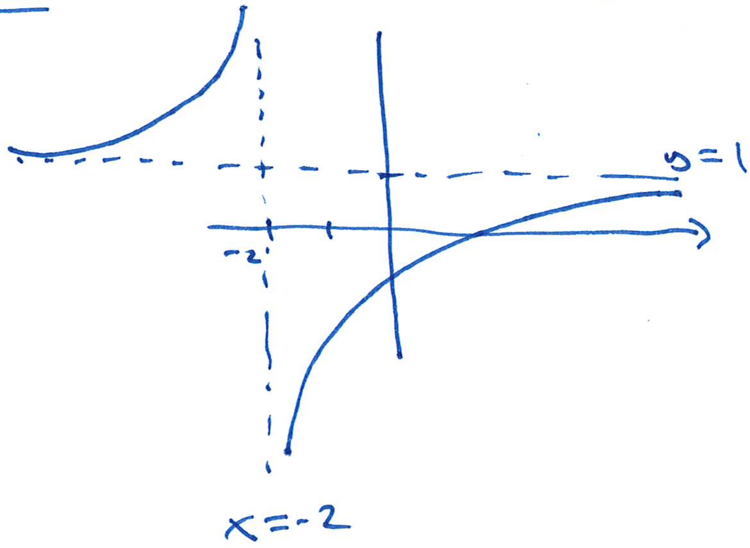
Ex: $\left(\dfrac{x}{x+2}\right)' = \dfrac{1\cdot(\cancel{x}+2) - \cancel{x}\cdot 1}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

$f(x) = \dfrac{x}{x+2}, \quad x \neq -2$

$x \to -2: f \to \pm\infty$

$x \to \pm\infty: f = \dfrac{x}{x+2} \dfrac{\cdot \frac{1}{x}}{\cdot \frac{1}{x}} = \dfrac{1}{1 + 2/x}$



Ex: $f = \sqrt{x^2+1} = \sqrt{u}, \quad u = x^2+1$

$= u^{1/2}$, $\quad$ —"—

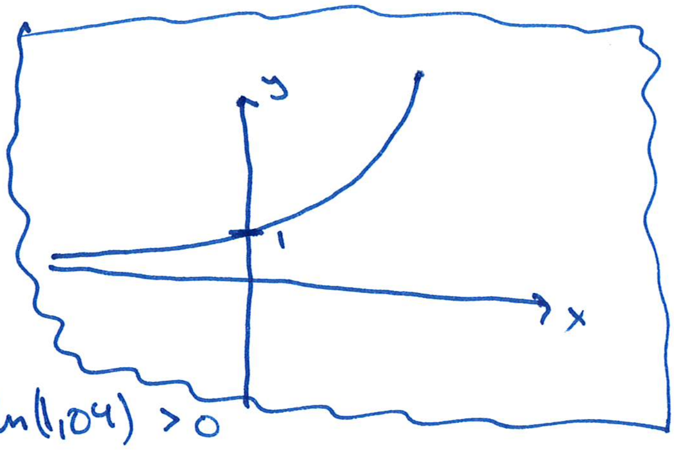
outer fn. $\qquad$ inner fn. (kernel)

$f' = \underbrace{\tfrac{1}{\cancel{2}} u^{-1/2}}_{\text{der. of } \sqrt{u}} \cdot \underbrace{\cancel{2}x}_{\text{der. of kernel } x^2+1}$

$= \dfrac{x}{u^{1/2}} = \dfrac{x}{\sqrt{u}} = \dfrac{x}{\sqrt{x^2+1}}$

(2) Exponential functions

$f(x) = a^x \quad (a > 0)$



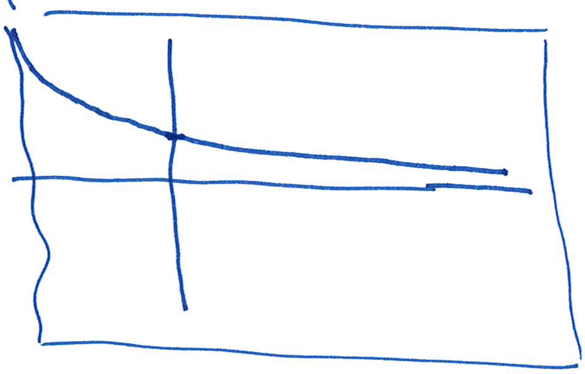
Ex: $f(x) = 1.04^x$

$f(0) = 1$ $\qquad f'(x) = 1{,}04^x \cdot \ln(1{,}04) > 0$

$f(1) = 1{,}04$ $\qquad \Rightarrow f$ is increasing

$f(2) = 1{,}04^2$

$x \to -\infty: \; \underset{1{,}04^x}{\overset{\|}{f(x)}} \to 0 \qquad x \to \infty: \; f(x) \to \infty$



Ex: $f(x) = 0.90^x$

$f'(x) = 0.90^x \cdot \ln(0.90) < 0$

decreasing fn.

$\underline{f(x) = a^x}$:

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot (a^h - 1)}{h} = a^x \cdot \underbrace{\left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)}_{\ln(a)}$$

Logarithms:

$\log_a(x)$ is the inverse function of a^x $(a>0)$

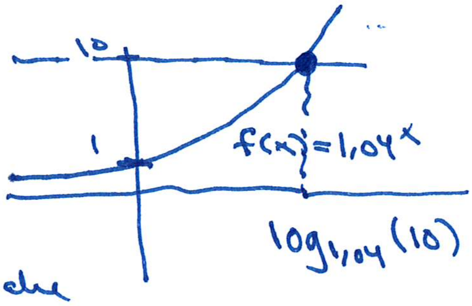
Fact:

When $a=e$ then $\lim_{h\to 0} \frac{a^h-1}{h} = 1$

$\boxed{e = 2.71828\ldots \text{ Eulers number}}$

$(e^x)' = e^x$

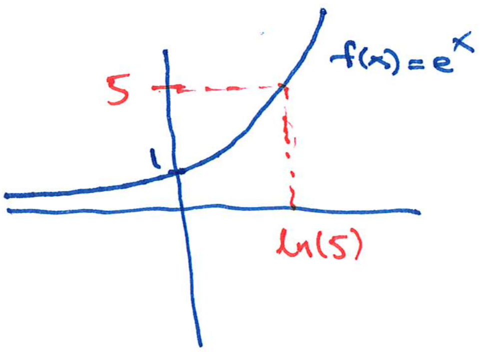
Ex: $1{,}04^x = 10$



"$\log_{1{,}04}(10)$ is the x-value such that $1{,}04^x = 10$"

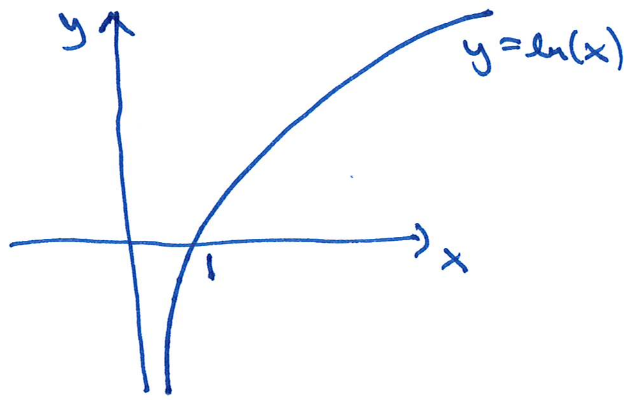
When $a=e$, then $\log_e(x) = \ln(x)$ is the natural logarithm

$f(x) = e^x$



$f'(x) = e^x > 0$ (f inc.)

$f(x) = \ln(x)$



$f(x) = \ln x$ defined for $x>0$

$f'(x) = 1/x > 0$ (f inc.)

$\ln(x) < 0$ $(x<1)$ $\ln(1) = 0$ $\ln(x) > 0$ $(x>1)$

Ex: $2^x = 100 \quad | \ln(\cdot)$

$$\ln(2^x) = \ln(100)$$

$$x \cdot \ln(2) = \ln(100)$$

$$\underline{\underline{x = \frac{\ln(100)}{\ln(2)}}}$$

Rules for logarithms:

i) $\ln(ab) = \ln(a) + \ln(b)$

ii) $\ln(a/b) = \ln(a) - \ln(b)$

iii) $\ln(a^n) = n \ln(a)$

Derivation rules: $\boxed{(e^x)' = e^x \qquad (\ln x)' = 1/x}$

Ex: $f(x) = e^{\sqrt{x}} = e^u, \quad u = \sqrt{x} = x^{1/2}$

$$f'(x) = e^u \cdot \tfrac{1}{2} x^{-1/2} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \underline{\underline{\frac{e^{\sqrt{x}}}{2\sqrt{x}}}}$$

$$f(x) = \ln(x^2+1) = \ln(u), \quad u = x^2+1$$

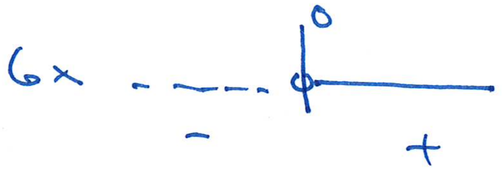
$$f'(x) = \frac{1}{u} \cdot 2x = \underline{\underline{\frac{2x}{x^2+1}}}$$

3 Higher derivatives and convexity

Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = \underline{3x^2 - 3}$

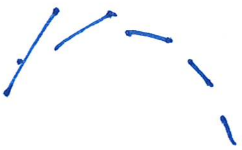
$f''(x) = (3x^2-3)' = 3 \cdot 2x - 0 = \underline{6x}$

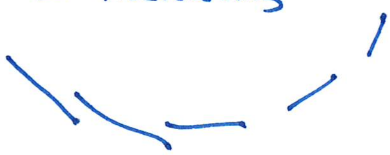


f is convex in $[0, \rightarrow)$
f is concave in $(\leftarrow, 0]$

Slope of the tangent line is decreasing

Slope of the tangent line is increasing



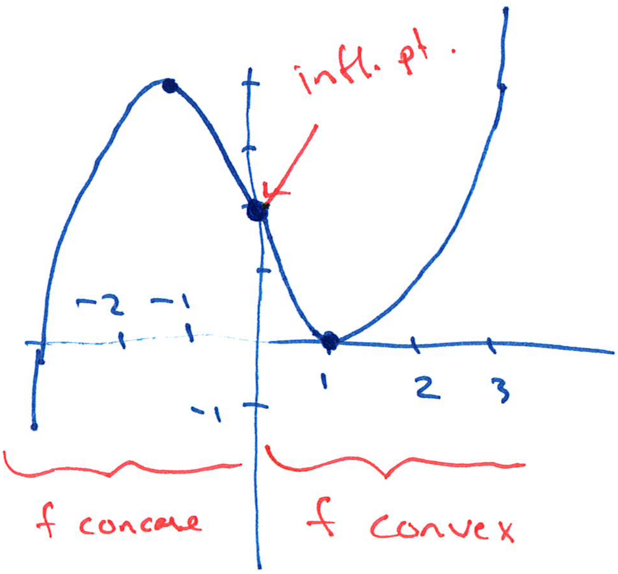


f is <u>concave</u> when $f''(x) \le 0$

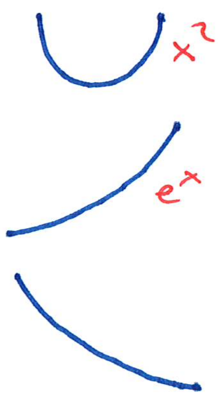
f is <u>convex</u> when $f''(x) \ge 0$

When $f''(x) = 0$, x is called an <u>inflexion pt</u>

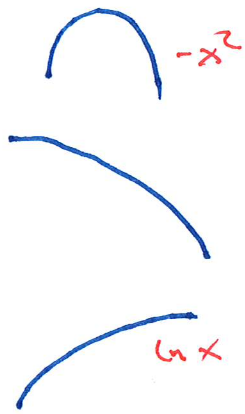
Ex:



<u>Convex:</u>



<u>Concave:</u>



Convex optimalization: max/min $f(x)$

Result:

If f is convex (everywhere), then any local min is a global min.
(stationary pt)

If f is concave (everywhere) then any stat. pt. is a global max

Solve locally:

find local max/min

i) Find stationary pts:
$f'(x) = 0$ (solve)

ii) Figure out if the stationary pts are local max/min

(use sign diagram for $f'(x)$)

Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$, $0 \leq x < 1$

$$f'(x) = \frac{3}{5}\cdot\frac{1}{1+x}\cdot 1 + \frac{2}{5}\cdot\frac{1}{1-x}\cdot(-1)$$

$$= \frac{3/5}{1+x} - \frac{2/5}{1-x} = \frac{3/5(1-x) - 2/5(1+x)}{(1+x)(1-x)}$$

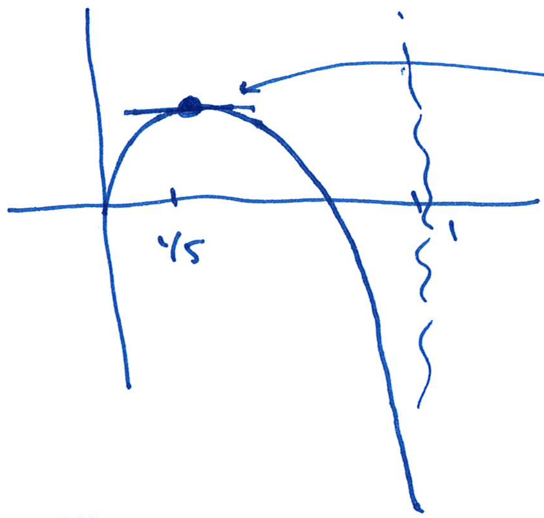
$$= \frac{1/5 - x}{(1+x)(1-x)} = 0 \qquad \underline{x = 1/5} \text{ stat. pt.}$$

$$f''(x) = \left(\frac{3/5}{1+x} - \frac{2/5}{1-x}\right)' = \left(3/5\cdot(1+x)^{-1} - 2/5\cdot(1-x)^{-1}\right)'$$

$$= 3/5\,(-1)(1+x)^{-2}\cdot 1 - 2/5\cdot(-1)(1-x)^{-2}\cdot(-1)$$

$$= \frac{-3/5}{(1+x)^2} + \frac{-2/5}{(1-x)^2} < 0 \qquad f \text{ \underline{concave}}$$

$\Rightarrow x = 1/5$ is max



We participate in a game where you win an amount equal to your bet size with probability 3/5, and loose a similar amount with probability 2/5. If we bet the constant share x of our capital in each game, then $f(x)$ is the expected value of the "growth rate"

$$f(x) = E\left[\ln(x_n/x_0)^{1/n}\right]$$

of the capital x_n after n games. Optimal $x = 1/5$ (Kelly bet).

Eivind Eriksen, Office B3-085y, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/FORK1003/