Plan

(1) Introduction

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(2) Solving linear systems of equations

Ex:

$$5x+7y=8$$
$$3x-8y=17$$

2×2 linear system
" "
2 eqns. 2 variables (unknowns)

$$x+y+z+w=10$$
$$2x+3y-w=12$$
$$5x-y+3z-w=9$$

3×4 linear system

Linear equations = polynomial equations of the first degree

Solution methods
- substitution methods
- elimination methods

Ex:

$$5x+7y=8$$
$$3x-8y=17$$

Substitution:

$$5x+7y=8$$
$$\frac{5x}{5}=\frac{8-7y}{5}$$
$$x=8/5-7/5\cdot y$$

$$3x - 8y = 17$$

$$3 \cdot \frac{8-7y}{5} - 8y = 17 \quad | \cdot 5$$

$$3(8-7y) - 40y = 85$$

$$-21y - 40y = 85 - 24$$

$$-61y = 61$$

$$\underline{y = -1}$$

$$x = \frac{8-7y}{5} = \frac{8-7\cdot(-1)}{5} = 3$$

$$\underline{x = 3}$$

Solution: $(x,y) = \underline{\underline{(3,-1)}}$

Elimination:

$$5x + 7y = 8 \quad | \cdot 3$$

$$3x - 8y = 17 \quad \cdot 5$$

$$\text{I} \quad 15x + 21y = 24$$

$$\text{II} \quad 15x - 40y = 85$$

$$\text{I} \quad 15x + 21y = 24$$

$$\text{II} - \text{I} \quad -61y = 61$$

$$\underline{y = -1}$$

$$15x - 21 \overset{\text{I}}{=} 24 \Rightarrow 15x = 45 \Rightarrow \underline{x = 3}$$

Solution: $(x,y) = \underline{\underline{(3,-1)}}$

(3) Key method: Gaussian elimination

Ex:

(i)
$$\begin{aligned} x+y+z+w &= 10 \\ 2x+3y \quad -w &= 12 \\ 5x-y+3z-w &= 17 \end{aligned}$$

3×4

(ii)
$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ 2x+3y+6z &= 12 \end{aligned}$$

3×3

a) Find the augmented (extended) coeff. matrix:

(ii)
$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ 2x+3y+6z &= 12 \end{aligned} \rightsquigarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 6 & 12 \end{array}\right) \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ x & y & z & \end{array} \end{array}$$

3×3 lin. system in standard form

b) Use elementary row operations until the matrix is in echelon form

Ex:
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 6 & 12 \end{array}\right) \begin{matrix} -1 \\ + \\ \end{matrix} \qquad (-1\ \ -1\ \ -1 \mid -3)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 6 & 12 \end{array}\right) \begin{matrix} -2 \\ \\ + \end{matrix} \qquad (-2\ \ -2\ \ -2 \mid -6)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 4 & 6 \end{array}\right)$$

Elementary row operations:

i) Switch two rows

ii) Multiply a row by $c \neq 0$

iii) Add a multiple of one row to another row.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 4 & 6 \end{array}\right) \begin{array}{l} \\ ]\,\cdot(-1) \\ \leftarrow + \end{array} \qquad (0 \;\; -1 \;\; -3 \mid -4)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

echelon form

Echelon form:

The first non-zero number in a row is called a pivot.

A matrix is in echelon form if:

i) If the matrix has zero rows, then they are on the bottom of the matrix.

ii) Each pivot are further to the right than the pivots in the rows above.

Procedure:

- Start with the upper left corner (if the coefficient is zero, switch with another row to make it non-zero)
  $\Rightarrow$ first pivot
- make all entries under the pivot zero (using the pivot and elem. row operations of type III)
- move to the next row, etc.

In general:

- Any matrix can be transformed to an echelon form using elementary row operations.
- The echelon form is not unique.

c) Use back substitution to solve the system.

Ex:

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right) \rightsquigarrow \begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ z &= 2 \end{aligned} \quad \begin{aligned} x &= 3 \\ y &= -2 \\ z &= 2 \end{aligned}$$

echelon form

$z = 2$

$y + 3z = 4$

$y = 4 - 3z = 4 - 3 \cdot 2 = -2$

$x + y + z = 3$

$x = 3 - y - z = 3 + 2 - 2 = 3$

Solution:

$(x, y, z) = (3, -2, 2)$

In general:

An echelon form is not unique, but the pivot positions are the same in two echelon forms.

Pivot position = positions in the echelon form that contains pivots.

In this case: (1,1) (2,2) (3,3)

position in row 1, col 1; row 2 col 2; row 3 col 3

Ex.:
$$\begin{aligned} x+y+z+w &= 10 \\ 2x+3y\qquad -w &= 12 \\ 5x-y+3z-w &= 17 \end{aligned}$$

3x4 lin. syst.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 2 & 3 & 0 & -1 & 12 \\ 5 & -1 & 3 & -1 & 17 \end{array}\right) \begin{array}{l} -2 \\ -5 \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & -2 & -3 & -8 \\ 0 & -6 & -2 & -6 & -33 \end{array}\right) 6 \rightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & -2 & -3 & -8 \\ 0 & 0 & -14 & -24 & -81 \end{array}\right) \end{array}$$

echelon form

Alt. $\cdot\left(-\frac{1}{14}\right)$

$$\begin{aligned} x+y+z+w &= 10 \\ y-2z-3w &= -8 \\ -14z-24w &= -81 \end{aligned}$$

$$-14z-24w=-81$$

$$\frac{-14z}{-14}=\frac{-81+24w}{-14}$$

$$z=\frac{81}{14}-\frac{24}{14}w=\frac{81}{14}-\frac{12}{7}w$$

$$y-2z-3w=-8$$
$$y=2z+3w-8$$
$$=2\left(\frac{81}{14}-\frac{24}{14}w\right)+3w-8$$
$$=\frac{81}{7}-8-\frac{24}{7}w+3w$$
$$=\frac{25}{7}-\frac{3}{7}w$$

$$x=10-y-z-w$$
$$=10-\left(\frac{25}{7}-\frac{3}{7}w\right)-\left(\frac{81}{14}-\frac{24}{14}w\right)-w$$
$$=\frac{70-25}{7}-\frac{81}{14}+\frac{3}{7}w+\frac{12}{7}w-\frac{7}{7}w$$
$$=\frac{90-81}{14}+\frac{3+12-7}{7}w=\frac{9}{14}+\frac{8}{7}w$$

Solutions: $(x,y,z,w)=\left(\frac{9}{14}+\frac{8}{7}w,\ \frac{25}{7}-\frac{3}{7}w,\ \frac{81}{14}-\frac{12}{7}w,\ w\right)$

where w is a free variable

In general: A variable is basic if there is a pivot position in the corresponding column.

A variable is free if there is not a pivot position in the corr. column.

$\Rightarrow$ We can solve for the basic variables, and get expressions in terms of the free var's

+

The free variables can have any values.

Ex.: x, y, z basic
w free

Consistent case:

* pivot positions in all variable columns (no free variables) $\Rightarrow$ one solution

* at least one variable column has no pivot position (at least one var. is free) $\Rightarrow$ infinitely many solutions

Inconsistent case:

* no solutions

Thm: Any linear system has either

- no solutions } inconsistent
- one unique solution
- infinitely many solutions } consistent

Moreover, you can distinguish these three cases when you know the pivot positions.

Ex:

$$\begin{aligned} x+y+2z&=7\\ 2x-y+3z&=1\\ 3x\quad+5z&=6 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7\\ 2 & -1 & 3 & 1\\ 3 & 0 & 5 & 6 \end{array}\right) \begin{array}{l} -2\\ -3 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7\\ 0 & -3 & -1 & -13\\ 0 & -3 & -1 & -15 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7\\ 0 & -3 & -1 & -13\\ 0 & 0 & 0 & -2 \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+2z&=7\\ -3y-z&=-13\\ -3y-z&=-15 \end{aligned}$$

no solution

$$\begin{aligned} x+y+2z&=7\\ -3y-z&=-13\\ 0&=-2 \end{aligned}$$

not possible

$\Downarrow$

no solutions

In general: A linear system has no solutions

$\Updownarrow$

There is a pivot pos. in the last column

Defn.: $\boxed{\text{A linear system is called } \underline{\text{inconsistent}} \text{ if there are no solutions, and } \underline{\text{consistent}} \text{ otherwise.}}$

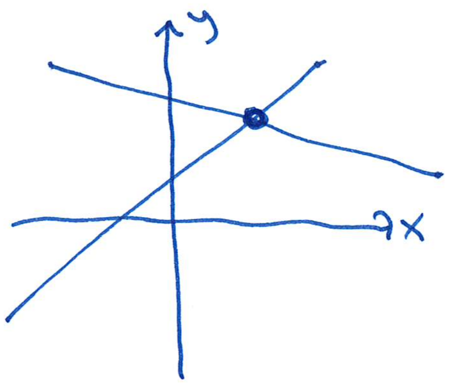
inconsistent $\Longleftrightarrow$ pivot pos. in the last column

consistent $\Longleftrightarrow$ no pivot pos. — " —

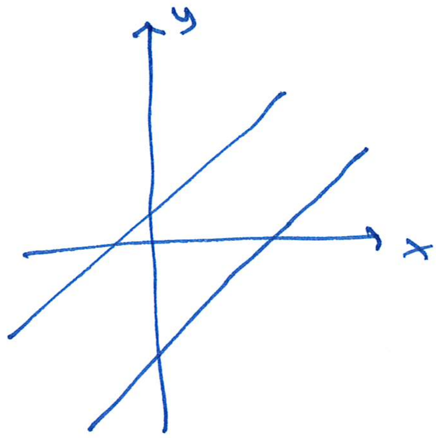
Geometry: 2 x 2 case

$$\boxed{\begin{aligned} ax+by&=c \\ dx+ey&=f \end{aligned}}$$

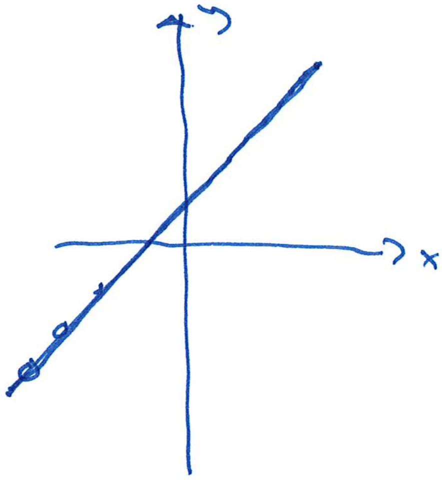
each equation gives a line in 2-dim coord. sys.



one solution



no solutions



inf. many solutions