Plan

1 Vectors and matrices
2 Matrix multiplication
3 Inverse matrices

① Vectors and matrices

Defn: An $m \times n$ matrix A is a rectangular array of numbers (m rows, n cols)

Ex: $A = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ 2×2 matrix

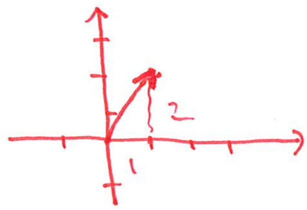
$B = \begin{pmatrix} 3 & 7 & 1 \\ 4 & 1 & 2 \end{pmatrix}$ 2×3 matrix

Defn: An n-vector $\underline{v}$ $= \vec{v}$ is an $n \times 1$-matrix (column vectors)

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2-vector

$\underline{w} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ 3-vector

Geometric representation



$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Length: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

length of the vector $\underline{v}$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\|\underline{v}\| = \sqrt{1^2 + 2^2}$
$= \sqrt{5}$

Vector: something with magnitude (length) and direction

Operations:

Addition: $A+B$

Ex: $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 5 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 1 & 7 \end{pmatrix}$

defined when A and B have the same size

Subtraction: $A-B$

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 7 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 3 & 3 \\ 4 & -1 & -3 \end{pmatrix}$

Scalar multiplication: $r \cdot A$

scalar = number

Ex: $2 \cdot \begin{pmatrix} 1 & 4 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 8 & -2 \end{pmatrix}$

Vector addition: $\underline{v} + \underline{w}$

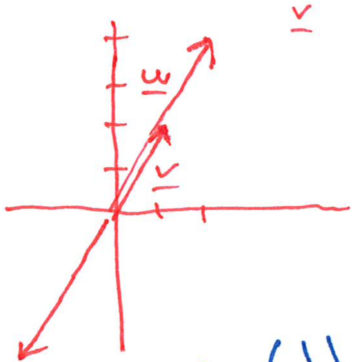
Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Vector subtraction: $\underline{v} - \underline{w}$

Ex: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

Vectors: $r \cdot \underline{v}$

Ex: $2 \cdot \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{= \underline{v}} = \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{= \underline{w}}$



$-2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = -\underline{w}$

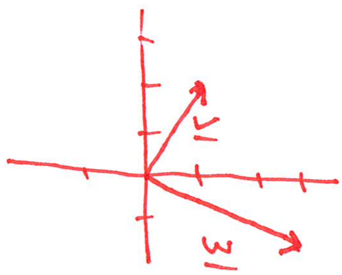
<u>Inner products of vectors</u> : $\underline{v} \cdot \underline{w}$ defined when $\underline{v}, \underline{w}$ have the same size

(dot product) (scalar produkt)

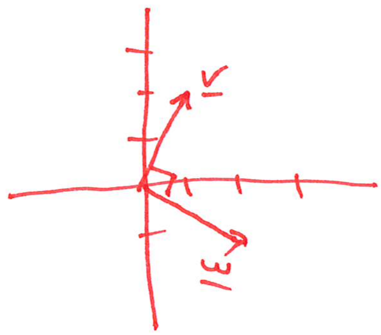
<u>Ex</u>: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 1 \cdot 3 + 2 \cdot (-1)$
$= 3 - 2 = \underline{1}$

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 1 \cdot 2 + 2 \cdot (-1)$
$= 2 - 2 = \underline{0}$

<u>Result</u>: $\underline{v} \cdot \underline{w} = 0 \iff \underline{v}$ and $\underline{w}$ are perpendicular
$(\underline{v} \perp \underline{w})$



$\underline{v} \cdot \underline{w} = 1$



$\underline{v} \cdot \underline{w} = 0$

② Matrix multiplication

$A \cdot B$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = ?$

$2 \times 2 = 2 \times 2$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 11 & 5 \\ 10 & 4 \end{pmatrix}}}$

(a) Matrix mult $A \cdot B$ is defined if #cols in A = #rows in B.

(b) If A is $m \times n$ and B is $n \times p$, then $A \cdot B$ is an $m \times p$ matrix.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4 = 11$ $\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 2 = 5$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2 \cdot 3 + 1 \cdot 4 = 10$ $\quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \cdot 1 + 1 \cdot 2 = 4$

Remarks:
i) Matrix multiplication is <u>not</u> position by position
ii) $A \cdot B \neq B \cdot A$ (matrix multiplication is noncommutative)

Ex: $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 5 & 7 \\ 8 & 10 \end{pmatrix}}} \neq \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

$2 \times 2 = 2 \times 2$

Ex: $\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the <u>identity matrix</u>

$\boxed{\begin{array}{l} A \cdot I = A \\ I \cdot A = A \end{array}}$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is also called the identity matrix

$A \cdot I = A$

$I \cdot A = A$

Connection to linear systems

Ex:

$$\begin{aligned} x+y+z+w &= 4 \\ x-y+z-w &= 0 \\ x+y-z-w &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 1 & 1 & 1 & | & 4 \\ 1 & -1 & 1 & -1 & | & 0 \\ 1 & 1 & -1 & -1 & | & 0 \end{pmatrix} = (A|\underline{b})$$

augmented matrix

$\updownarrow$

Gaussian elimination

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

coeff. matrix

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$A \cdot \underline{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z+w \\ x-y+z-w \\ x+y-z-w \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \underline{b}$$

Matrix form of a linear system: $\boxed{A\underline{x} = \underline{b}}$

Connection with the dot/inner product:

Ex: $\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 1\cdot 2 + 2\cdot 3 = \underline{8}$ dot product of vectors

$\begin{pmatrix}1 & 2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = (1\cdot 2 + 2\cdot 3) = (8)$ matrix multiplication

$1\times 2 = 2\times 1$

Transpose: $A \rightsquigarrow A^T$

A: $m\times n$ matrix, A^T: $n\times m$ matrix

Ex: $A = \begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\end{pmatrix}$ 2×3 $\quad A^T = \begin{pmatrix}1 & 0\\2 & 1\\3 & 0\end{pmatrix}$ 3×2

$A = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} \quad A^T = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$

$A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} \quad A^T = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$

$A = \begin{pmatrix}1 & 2 & 3\\2 & 0 & 1\\3 & 1 & 4\end{pmatrix}$ 3×3 symmetric matrix

Defn: A matrix is symmetric if $A^T = A$.

Remark: If $A = \underline{v}$ and $B = \underline{w}$ are $m\times 1$ and $n\times 1$ matrices and m- and n-vectors.

$A = \underline{v} = \begin{pmatrix}v_1\\v_2\\\vdots\\v_m\end{pmatrix} \quad B = \underline{w} = \begin{pmatrix}w_1\\w_2\\\vdots\\w_n\end{pmatrix}$

$\underline{v}\cdot\underline{w} = A^T\cdot B$

dot product: $v_1w_1 + v_2w_2 + \dots$

matrix mult.: $\begin{pmatrix}v_1 & v_2 & \dots\end{pmatrix}\begin{pmatrix}w_1\\w_2\\\vdots\end{pmatrix} = (v_1w_1 + \dots)$

Square matrices:

An $m \times n$ matrix is called square if $m = n$

Note: * If a matrix is symmetric, it must be square

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$$

* The identity matrix is square $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

* If A is a square matrix $(n \times n)$, then we can compute powers of A

$$A^2 = A \cdot A$$
$$A^3 = A \cdot A \cdot A$$
$$A^4 = A \cdot A \cdot A \cdot A$$
$$\vdots$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}$$

$$\vdots$$

$$A^n = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$ Hard to compute A^n!

$\frac{1}{2}(3^n + (-1)^n)$

3 Inverse matrices

$$3x = 6 \quad \Big| \cdot \tfrac{1}{3} = 3^{-1}$$
$$3^{-1}\cdot 3 = 1$$
$$\cancel{3^{-1}}\cdot\cancel{3}x = 3^{-1}\cdot 6$$
$$A\cdot \underline{x} = \underline{b}$$

A $n\times n$ matrix

Defn: An inverse of A is a matrix A^{-1} such that

$$A\cdot A^{-1} = I$$
$$A^{-1}\cdot A = I$$

Note: * only square matrices can have an inverse, and not all square matrices have an inverse.

* if A has an inverse, then it is unique

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $n=2$

$ad-bc=0$: A has no inverse

$ad-bc\neq 0$: A has an inverse

$$A^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$ad-bc = 1\cdot 4 - 2\cdot 3 = 4-6 = -2 \neq 0$$

$$A^{-1} = \frac{1}{-2}\cdot\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \underline{\frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}}$$

$$A^{-1}\cdot A = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A\cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex: $x + 2y = 7$
$3x + 4y = 11$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b} \quad | \; A^{-1} \cdot$$
$$A^{-1} A \, \underline{x} = A^{-1} \underline{b}$$
$$I \cdot \underline{x} = A^{-1} \underline{b}$$
$$\underline{x} = A^{-1} \underline{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 7 \\ 11 \end{pmatrix}}_{\underline{b}}$$

$$= \frac{1}{2} \begin{pmatrix} -6 \\ 10 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -3 \\ 5 \end{pmatrix}}}$$

Defn: A square matrix is called <u>invertible</u> if A^{-1} exists, and <u>non-invertible</u> otherwise.

Result: If a linear system has matrix form $A \cdot \underline{x} = \underline{b}$ where A is a square invertible matrix, then the linear system has one unique solution $\underline{x} = A^{-1} \cdot \underline{b}$.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad ad - bc = 1 \cdot 4 - 2 \cdot 2 = 0$

$\Rightarrow A$ is <u>not invertible</u>

Results: A $n \times n$-matrix

i) A is invertible $\iff$ $|A| \neq 0$

determinant of A

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

ii) If $|A| \neq 0$, then $\boxed{A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$