Plan

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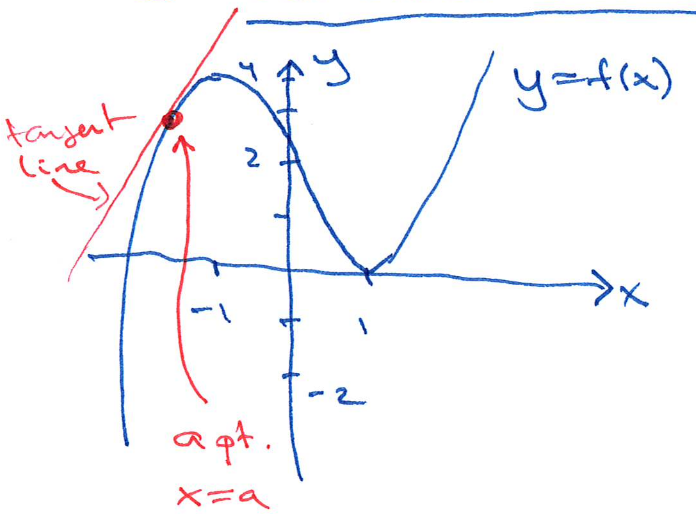
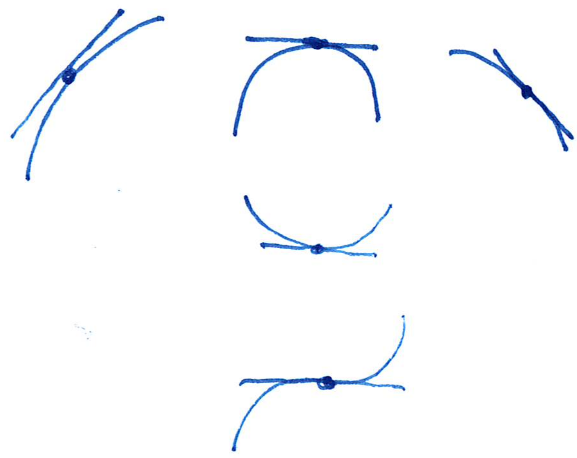
① Functions and derivatives



Defn: The derivative $f'(a)$ is the slope of the tangent line of $y=f(x)$ at $x=a$

$f'(a)>0$ $f'(a)=0$ $f'(a)<0$



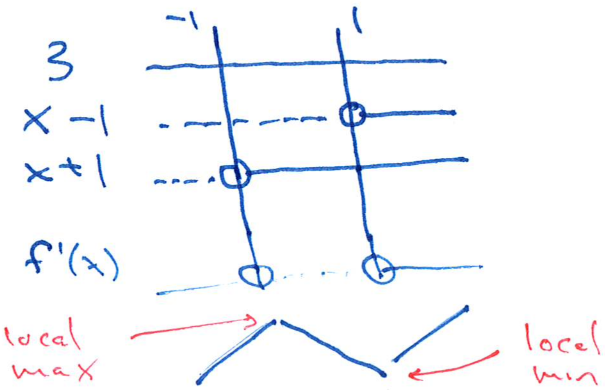
Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$x=-2$: $f'(-2) = 3(-2)^2 - 3 = \underline{9}$

$f'(x) = 3x^2 - 3 = 3(x^2-1)$

$= 3(x-1)(x+1)$



Computing derivatives:

Derivation rules:

i) $(x^n)' = n x^{n-1}$ (any const. n)

ii) $(u \pm v)' = u' \pm v'$ (any expr. u, v)

iii) $(c \cdot u)' = c \cdot u'$ (c const., u expr)

iv) $(u \cdot v)' = u'v + uv'$ (u, v any expr)

v) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ — " —

Ex: $\left(\frac{x}{x+2}\right)' = \frac{1 \cdot (\cancel{x}+2) - \cancel{x \cdot 1}}{(x+2)^2} = \underline{\underline{\frac{2}{(x+2)^2}}}$

$f(x) = \frac{x}{x+2} \qquad f'(x) = \frac{2}{(x+2)^2} \qquad f'(1) = \frac{2}{3^2} = 2/9 > 0$

vi) Chain rule: $f(x) = h(u(x)) \Rightarrow f'(x) = h'(u(x)) \cdot u'(x)$

Ex: $f(x) = \sqrt{1-x^2}$

$h(u) = \sqrt{u} = u^{1/2} \quad h'(u) = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}}$

$u(x) = 1 - x^2 \quad u'(x) = 0 - 2x = -2x$

$f'(x) = \underbrace{\frac{1}{2\sqrt{1-x^2}}}_{h'(u(x))} \cdot \underbrace{(-2x)}_{u'(x)} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \underline{\underline{-\frac{x}{\sqrt{1-x^2}}}}$

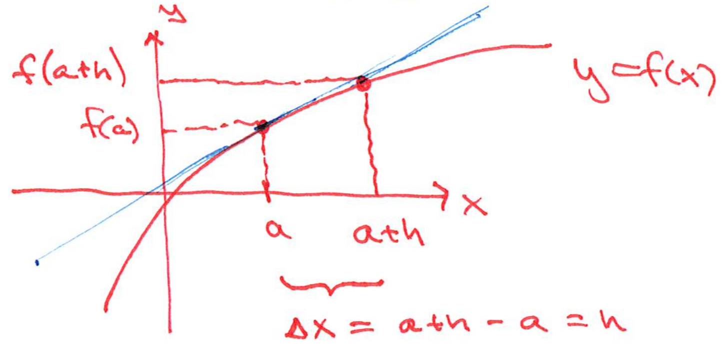
vii) Exponential fn. and logarithms:

$(e^x)' = e^x \qquad (\ln x)' = 1/x$

$(a^x)' = a^x \cdot \ln(a) \qquad (\log_a x)' = \frac{1}{x \cdot \ln(a)}$

Using the defn: $$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a) \leftarrow \Delta y}{h \leftarrow \Delta x}$$

$\Delta y = f(a+h) - f(a)$

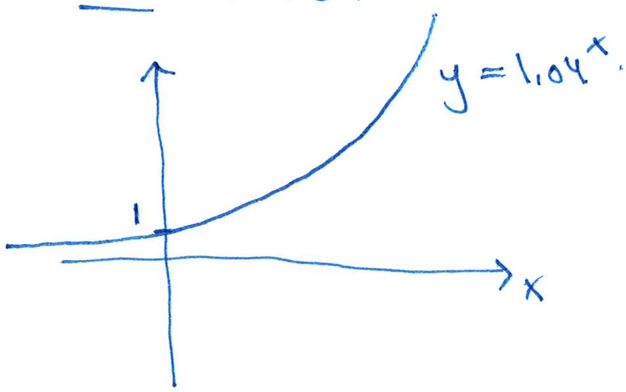


② Exponential functions and logarithms

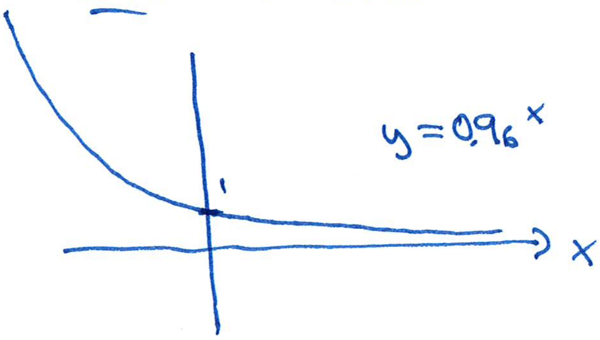
Exponential functions:

$f(x) = a^x$ (base a)

Ex: $a = 1.04$



Ex: $a = 0.96$



Note: $a > 0$ $a \neq 1$

Ex: ~~$a=-2$: $f(1/2) = (-2)^{1/2} = \sqrt{-2}$~~

$\Downarrow$

two cases: $\begin{cases} a > 1 \\ 0 < a < 1 \end{cases}$

Properties:

i) f is continuous and increasing / decreasing

ii) $D_f = \mathbb{R} = (-\infty, \infty)$

$V_f = (0, \to) = (0, \infty)$

iii) $f'(x) = a^x \cdot \ln(a)$ når $f(x) = a^x$

$f'(x) = e^x$ når $f(x) = e^x$

Euler's number: $e = 2.71828\ldots$

$\lim_{n\to\infty} (1 + 1/n)^n = e$ $\quad n=10: \left(1+\frac{1}{10}\right)^{10} = 1.1^{10} \simeq 2.59$

compound interest, cont. comp.

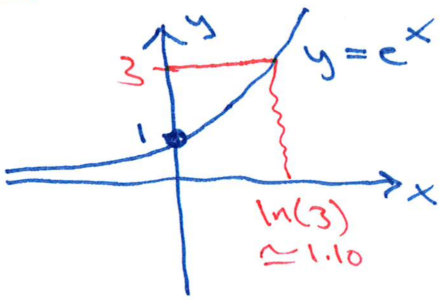
Ex: $f(x) = \frac{e^x+1}{e^x-1}$ $\quad f'(x) = \frac{e^x \cdot (e^x-1) - (e^x+1)\cdot e^x}{(e^x-1)^2}$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ $\quad \boxed{(e^x)' = e^x}$

$$= \frac{\cancel{(e^x)^2} - e^x - \cancel{(e^x)^2} - e^x}{(e^x-1)^2}$$

$$= \frac{-2e^x}{(e^x-1)^2} < 0$$

Ex: $f(x) = e^x$



$\ln(u)$: the value of x such that $e^x = u$

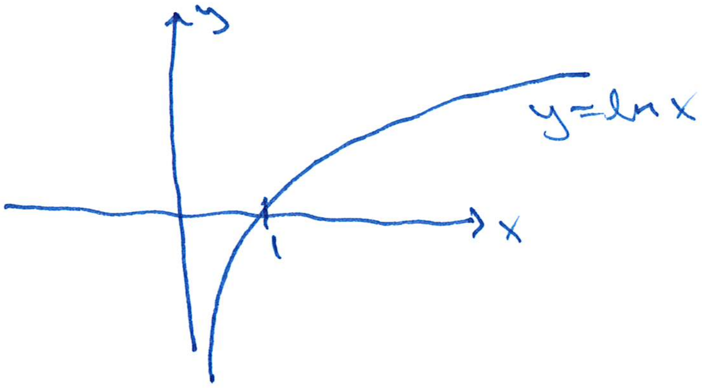
Logarithms: $f(x) = \log_a(x)$ (base $a > 0$, $a \neq 1$)

$f(x) = \log_e(x) = \ln(x)$

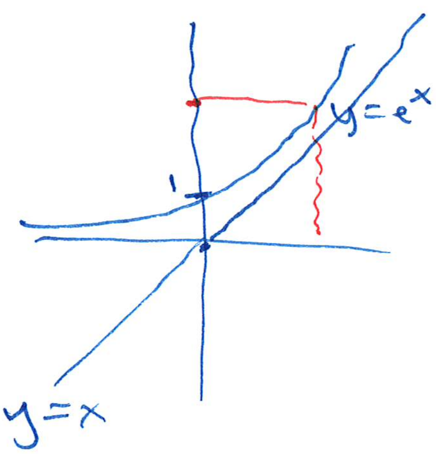
Defn: $\ln(x)$ is the inverse fn. of e^x

$\log_a(x)$ — " — a^x

Ex: $f(x) = \ln(x)$, $x > 0$



$f(x) = e^x$



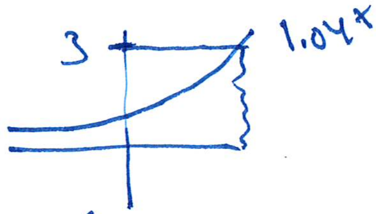
- f is increasing
- f is cont.
- $D_f = (0, \infty)$
- $V_f = (-\infty, \infty)$
- $(\ln x)' = 1/x$

$D_f = (-\infty, \infty)$

$V_f = (0, \infty)$

Ex:

$$\frac{100 \cdot 1{,}04^x}{100} = \frac{300}{100}$$



$$1{,}04^x = 3 \rightarrow x = \log_{1{,}04}(3)$$

$$\ln(1{,}04^x) = \ln 3$$

$$x \cdot \ln(1{,}04) = \ln(3)$$

$$x = \frac{\ln(3)}{\ln(1{,}04)}$$

$$\boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

Rules for logarithms:

i) $\ln(a \cdot b) = \ln(a) + \ln(b)$

ii) $\ln(a/b) = \ln(a) - \ln(b)$

iii) $\ln(a^n) = n \cdot \ln(a)$

3 Higher derivatives and convexity

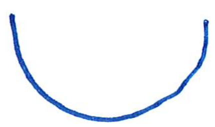
Ex: $f(x) = x^3 - 3x + 2$

$$f'(x) = \underline{3x^2 - 3}$$

$$f''(x) = (3x^2-3)' = 3 \cdot 2x - 0 = \underline{6x}$$

Interpretation:

$f'' > 0$: f' increasing



(convex)

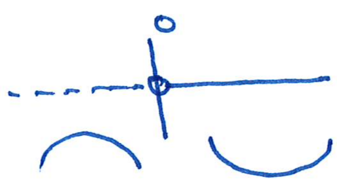
$f'' < 0$: f' decreasing

(concave)

Defn: f is convex on an interval I if $f''(x) \geq 0$ for all x in I

f is concave on an interval I if $f''(x) \leq 0$ for all x in I

Ex: $f'' = 6x$



concave convex

Convex optimization:

Defn: A pt where $f'(x) = 0$ is called a stationary pt.

Result: If f is convex (everywhere, on D_f), then any stationary pt. is a global min.

If f is concave (everywhere, on D_f), then any stationary pt is a global max.

Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$, $0 < x < 1$

$\boxed{\text{max/min } f(x)}$

$1+x>0$
$1-x>0$
$1>x$
$x>-1$

$\boxed{(\ln x)' = \frac{1}{x}}$

Stationary pts: $f'(x) = 0$

$$f'(x) = \frac{3}{5}\left[\ln(1+x)\right]' + \frac{2}{5}\left[\ln(1-x)\right]'$$

$$= \frac{3}{5}\frac{1}{x+1}\cdot 1 + \frac{2}{5}\cdot\frac{1}{1-x}\cdot(-1)$$

$$= \frac{3}{5}\cdot\frac{1\cdot(1-x)}{(x+1)\cdot(1-x)} + \frac{-2\cdot(1+x)}{5(1-x)\cdot(1+x)}$$

$$= \frac{3(1-x)-2(1+x)}{5(1+x)(1-x)} = \frac{1-5x}{5(1+x)(1-x)}$$

$f'(x)=0$: $\quad \frac{1-5x}{5(1+x)(1-x)} = 0$

$$1-5x=0$$

Stat. pt: $x = 1/5$

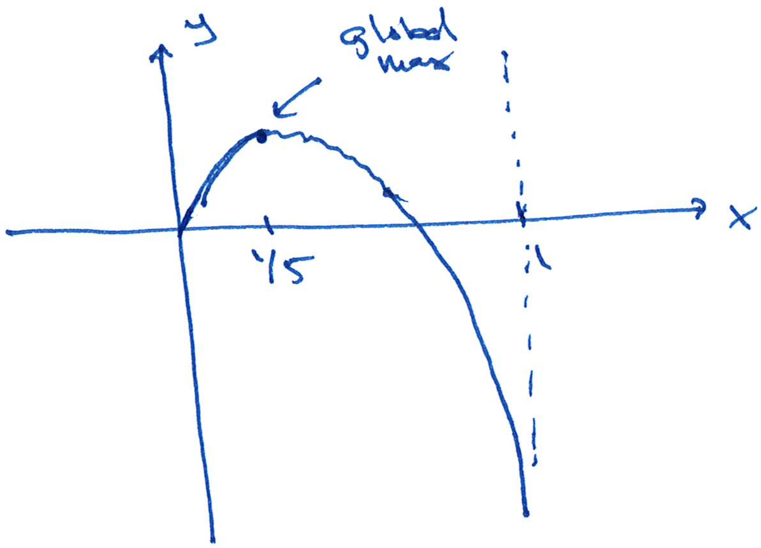
Convex/concave: $f''(x) = \left(\frac{3}{5}\cdot(x+1)^{-1} - \frac{2}{5}(1-x)^{-1}\right)'$

$$= \frac{3}{5}\cdot(-1)\cdot(x+1)^{-2}\cdot 1 - \frac{2}{5}\cdot(-1)\cdot(1-x)^{-2}\cdot(-1)$$

$$= -\frac{3}{5}\cdot\frac{1}{(x+1)^2} - \frac{2}{5}\frac{1}{(1-x)^2} < 0 \text{ for all } x.$$

$\Rightarrow$ f is concave on $D_f = (0,1)$

$\Rightarrow$ $x = 1/5$ is global max for f



$$f(x) = 0.6 \cdot \underline{\ln(1+x)} + 0.4\, \underline{\ln(1-x)}$$

$\underline{x \to 0:}$ $f(x) \to 0.6 \cdot \ln(1) + 0.4 \ln(1) = \ln(1) = 0$

$\underline{x \to 1:}$ $\left.\begin{array}{l} \ln(1+x) \to \ln(2) = 0.693\ldots \\ \ln(1-x) \to -\infty \end{array}\right\} f(x) \to -\infty$