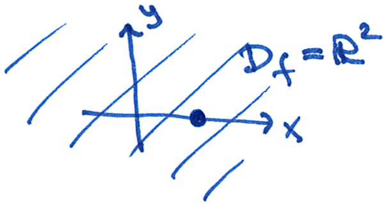
Plan

1. Functions in two variables and partial derivatives
2. Unconstrained optimization
3. Constrained optimization and Lagrange multipliers

1 Functions in two variables

Ex: $f(x,y) = x^3 - xy + y^2$ $\qquad f(1,0) = 1^3 - 1\cdot 0 + 0^2 = \underline{1}$

$D_f = \{(x,y) : - \} = \mathbb{R}^2$

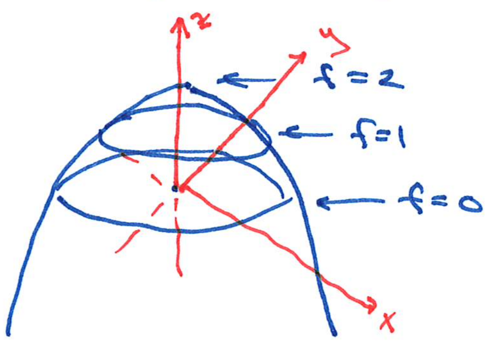


$z = f(x,y)$

Graph of f: (x,y,z) s.t. (x,y) in D_f, $z = f(x,y)$

$V_f = \{ z = f(x,y) : (x,y) \text{ in } D_f \}$ subset of $\mathbb{R}$

Ex: $f(x,y) = 2 - x^2 - y^2$



$V_f = \underline{(\leftarrow, 2]}$

Level curves of f

$f(x,y) = 0$:

$2 - x^2 - y^2 = 0$

$x^2 + y^2 = 2$

circle

$f(x,y) = 1$:

$2 - x^2 - y^2 = 1$

$x^2 + y^2 = 1$

$f(x,y) = 2$:

$x^2 + y^2 = 0$

Partial derivatives:

$$f'_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f'_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Ex: $f(x,y) = x^3 - xy + y^2$

$$f'_x(x,y) = 3x^2 - y\cdot 1 + 0 = \underline{3x^2-y} = f'_x = \frac{\partial f}{\partial x}$$

$$f'_y(x,y) = 0 - x\cdot 1 + 2y = \underline{-x+2y} = f'_y = \frac{\partial f}{\partial y}$$

② Unconstrained optimization

max/min $f(x,y)$

$$\boxed{\begin{aligned} f(1/6, 1/12) &= \left(\tfrac{1}{6}\right)^3 - \tfrac{1}{6}\cdot\tfrac{1}{12} + \left(\tfrac{1}{12}\right)^2 \\ &= \left(\tfrac{1}{6}\right)^3\cdot\left(1 - 3 + \tfrac{6}{4}\right) \\ &= -\tfrac{1}{2}\left(\tfrac{1}{6}\right)^3 < 0 \end{aligned}}$$

Stationary pts: $f'_x = f'_y = 0$ $\longleftarrow$ FOC = first order conditions

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y = 0$ $\qquad 3(2y)^2 - y = 0 \qquad 12y^2 - y = 0$

$f'_y = -x + 2y = 0 \quad \Rightarrow \underline{x = 2y} \qquad y(12y-1) = 0$

$y = 0$ or $y = 1/12$

$x = 2\cdot 0$, $x = 0$ | $x = 2\cdot 1/12 = 1/6$

Stat. pts: $(x,y) = \underline{(0,0)}, \underline{(1/6, 1/12)}$

$f = 0 \qquad f = -\frac{1}{2}\left(\frac{1}{6}\right)^3$

Fact: If (x,y) is a max or min for f, then it is either

i) a stationary pt.
ii) a pt where f'_x or f'_y does not exist
iii) a boundary pt for D_f

} Candidate pts for max/min.

Second derivative test:

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y \qquad f''_{xx} = 6x \qquad f''_{xy} = -1$

$f'_y = -x + 2y \qquad f''_{yx} = -1 \qquad f''_{yy} = 2$

$$H(f)(x,y) = \underline{\underline{\begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix}}} \longleftarrow \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

Hessian matrix

Second derivative test:

If (x^*,y^*) is a stationary pt of f, consider the matrix

$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{yx}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Then: If $\det H(f)(x^*,y^*) > 0$ and $A > 0$, then (x^*,y^*) local min.

— " — " $A < 0$, " " local max

If $\det H(f)(x^*,y^*) < 0$, then (x^*,y^*) is a saddle pt.

A stationary pt that is neither local min nor local max = saddle pt.

Note:

- If f is "nice", then $H(f)$ is symmetric.
- We have ~~det~~ $H(f)(x^*,y^*) = AC - B^2$
- If $\det H(f)(x^*,y^*) = 0$, the test is inconclusive.

Ex: $f(x,y) = x^3 - xy + y^2$

f has no max

f has no min

$f(-2,0) = -8$

$(0,0)$: saddle pt. $f = 0$

$$H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

$\det = 0 \cdot 2 - (-1)^2 = -1 < 0$

$(1/6, 1/12)$: local min $f = -\frac{1}{2}\left(\frac{1}{6}\right)^3 = -\frac{1}{432}$

$$H(f)(1/6, 1/12) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$\det = 1 \cdot 2 - (-1)^2 = 1 > 0 \qquad A = 1 > 0$

③ Constrained optimization problems or Lagrange multipliers

max / min $f(x,y)$ when ..... ← extra conditions on the pts (x,y) we are allowed to use

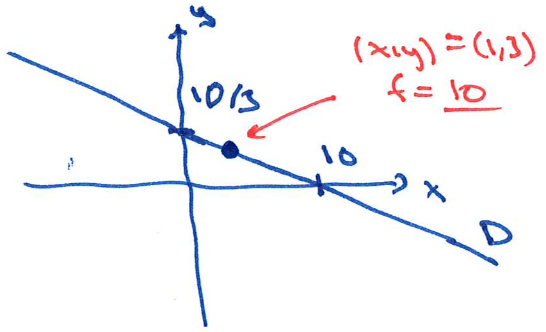
(eqn's / inequalities)

objective fn. $g(x,y) = a$

Ex: max / min $f(x,y) = x^2+y^2$ when $x+3y=10$

Lagrange problem: constraints are equations

Method: Lagrange multipliers



$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\,(g(x,y)-a)$$
$$= x^2+y^2-\lambda\cdot(x+3y-10)$$

$D = \{(x,y) : x+3y=10\}$ the set of adm. pts

$$\left.\begin{array}{l} \mathcal{L}'_x = 2x-\lambda(1) = 0 \\ \mathcal{L}'_y = 2y-\lambda(3)=0 \end{array}\right\} \text{FOC}$$
$$\left. x+3y = 10 \right\} C$$

Lagrange conditions

(1) $2x-\lambda=0$ $\quad x = \lambda/2$

(2) $2y-3\lambda=0$ $\quad y = 3\lambda/2$

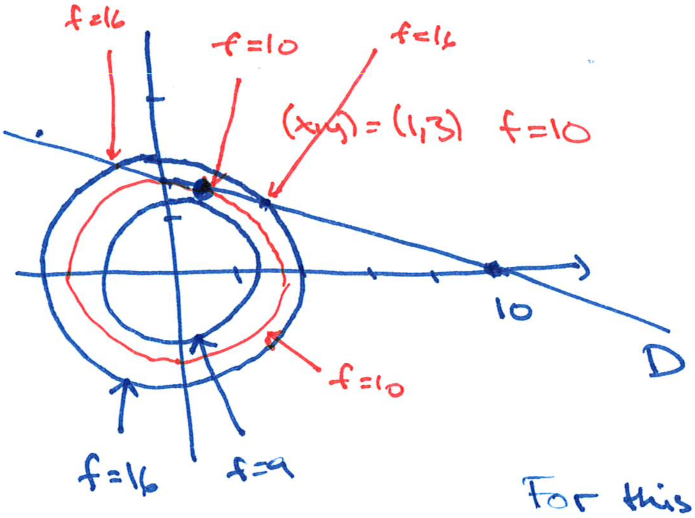
(3) $x+3y=10$ $\quad \lambda/2 + 3\cdot(3\lambda/2) = 10 \quad |\cdot 2$

$$\lambda + 9\lambda = 20$$
$$\frac{10\lambda}{10} = \frac{20}{10}$$

$\underline{\lambda=2}$ $\quad \underline{x=1}$ $\quad \underline{y=3}$

Candidate pts: $(x,y;\lambda) = (1,3;2)$

$f = 1^2+3^2 = \underline{10}$



Level curve: $f(x,y)=10$

$x^2+y^2=10 \quad r=\sqrt{10}$

$f(x,y)=9$: $x^2+y^2=9$, $r=3$

$f(x,y)=16$: $x^2+y^2=16$, $r=4$

For this example: $f_{min} = 10$ at $(x,y)=(1,3)$, $\lambda = 2$

there is no max for f on D

In general: The points $(x,y;\lambda)$ that satisfies the Lagrange conditions are exactly the points where the set D of adm. pts meets a ~~level~~ level curve of f at a tangent.

Interpretation of λ: marginal change of the max/min-value per unit change in a = constant in c

Ex: min $f(x,y)=x^2+y^2$ when $x+3y=10$ $\longrightarrow$ $f_{min}=10$

min $f(x,y)=x^2+y^2$ when $x+3y=11$ $\longrightarrow$ $f_{min} \simeq 10+2 = 12$

Extreme Value Thm:

If f is a continuous fn on a closed and bound set D, then f has a max and a min

Ex: max/min $f(x,y) = x+y$ when $x^3 - 3xy + y^3 = 0$

Lagrange:

$L = x + y - \lambda(x^3 - 3xy + y^3)$

$$\boxed{\begin{aligned} L'_x &= 1 - \lambda \cdot (3x^2 - 3y) = 0 \\ L'_y &= 1 - \lambda(-3x + 3y^2) = 0 \\ & x^3 - 3xy + y^3 = 0 \end{aligned}}$$

Cand. pt:

$(x, y; \lambda) = (3/2, 3/2; 4/9)$

$f = 3$

(1) $1 = 3\lambda(x^2 - y)$

(2) $1 = 3\lambda(-x + y^2)$

(3) $x^3 - 3xy + y^3 = 0$

$3\lambda = \frac{1}{x^2 - y}$

$3\lambda = \frac{1}{-x + y^2}$

$\Rightarrow \frac{1}{x^2 - y} = \frac{1}{y^2 - x}$

$y^2 - x = x^2 - y$

$y^2 - x^2 - x + y = 0$

$(y-x)(y+x) + (y-x) = 0$

$(y-x)(y+x+1) = 0$

$y = x$ or $x + y = -1 \rightarrow y = -1 - x$

$x^3 - 3x^2 + x^3 = 0$

$2x^3 - 3x^2 = 0$

$x^2(2x - 3) = 0$

~~$x = 0$, $y = 0$, $\lambda =$~~ or $x = 3/2$, $y = 3/2$, $\lambda = 4/9$

~~$x^3 - 3x(-1-x) + (-1-x)^3 = 0$~~

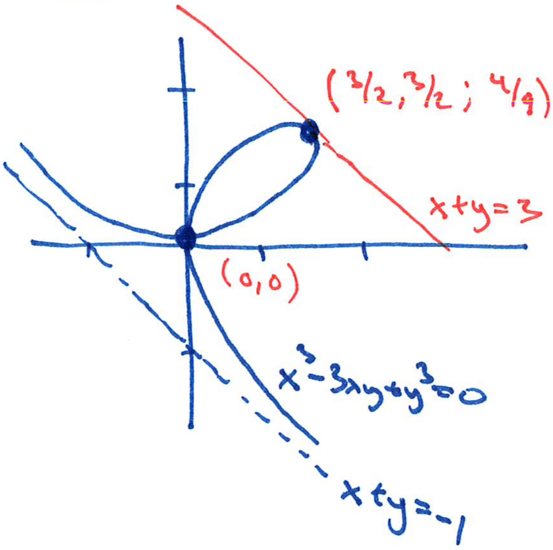
~~$x^3 + 3x + 3x^2 + (-1) - 3x - 3x^2 - x^3 = 0$~~

~~$-1 = 0$~~

$\lambda = \frac{1}{3} \cdot \frac{1}{x^2 - y}$

$= \frac{1}{3} \cdot \frac{1 \cdot 4}{\left(\frac{9}{4} - \frac{3}{2}\right) \cdot 4}$

$= \frac{4}{3 \cdot 3} = \frac{4}{9}$



max/min $f(x,y)=x+y$ when $x^3-3xy+y^3=0$

$V_f=(-1,3]$ $f_{max}=3$
f_{min} : no min