Plan

1. Key Method: Determinants
2. Linear systems, inverse matrices and determinants

① Determinants

$A \rightsquigarrow \det(A) = |A|$

$n \times n$ matrix ⟶ a number

$n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = ad - bc$

$n>2$: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

there is a formula for det(A), it is complicated (has $n!$ terms of degree n)

$n=3$: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$

$$\det(A) = aei + bfg + cdh - ceg - afh - bdi$$

Method: Cofactor expansion along the first row

signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot C_{11} + b \cdot C_{12} + c \cdot C_{13}$$

$$= a \cdot (+1) \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \cdot (-1) \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot (+1) \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= +a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$

M_{ij} = determinant of the submatrix you get by deleting row i and col. j.

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= 1(18-12) - 1(9-4) + 1(3-2) = 6-5+1 = \underline{\underline{2}}$

Fact: You can compute the determinant of any matrix using cofactor expansion along any row or column.

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$

$= -(9-4) + 2(9-1) - 3(4-1) = -5+16-9 = \underline{\underline{2}}$

Ex: $\begin{vmatrix} 2 & 14 & 73 \\ 0 & -1 & 147 \\ 0 & 0 & 3 \end{vmatrix} = +2 \cdot \begin{vmatrix} -1 & 147 \\ 0 & 3 \end{vmatrix} = +2 \cdot (-1) \cdot 3 = \underline{\underline{-6}}$

upper triangular matrix $\Rightarrow$ determinant is the product of diagonal entries

Fact: a square echelon form is upper triangular

$n > 3$: Cofactor expansion

Ex:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 1 & 1 & 4 & 2 \\ 2 & 1 & 0 & 7 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 2 & 1 & 0 \end{vmatrix} + 0 \cdot * - 2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 7 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= -1 \cdot \left(+3 \cdot (-1) - 4(4)\right) - 2\left(+2(4) - 1(1)\right) + 7\left(+1(-7) - 1(5) + 1 \cdot (3)\right)$$

$$= -1 \cdot (-19) - 2(7) + 7 \cdot (-9) = 19 - 14 - 63 = \underline{\underline{-58}}$$

Alternative method: Computing determinants using Gauss

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 1 & 1 & 4 & 2 \\ 2 & 1 & 0 & 7 \end{pmatrix} \begin{matrix} -2 \\ -1 \\ -2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & -1 & -2 & 5 \end{pmatrix} = B$$

Fact: Let $A \to B$ be an elementary row operation. Then:

i) If we switch two rows, then $|B| = -|A|$

ii) If we multiply a row with $c \neq 0$, then $|B| = c \cdot |A|$

iii) If we add a multiple of one row to another row, then $|B| = |A|$.

Alt a):

$$|A| = |B| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & -1 & -2 & 5 \end{vmatrix} = +1 \cdot \begin{vmatrix} -3 & 1 & -2 \\ 0 & 3 & 1 \\ -1 & -2 & 5 \end{vmatrix}$$

$$= +1\left(-3(17) - 1 \cdot (7)\right) = (-51 - 7) = \underline{\underline{-58}}$$

Alt b)

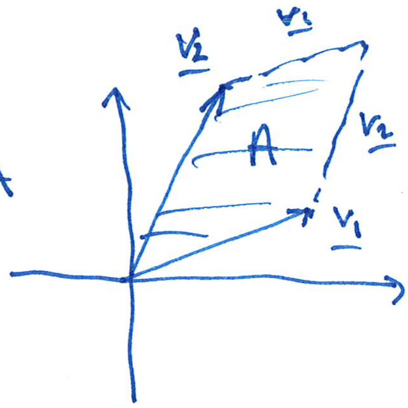
$$B = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & -1 & -2 & 5 \end{vmatrix} \cdot(-2) \to \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 5 & -12 \\ 0 & 0 & 3 & 1 \\ 0 & -1 & -2 & 5 \end{vmatrix} \cdot(-1)$$

$$\to \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 5 & -12 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -7 & 17 \end{vmatrix} \cdot 7/3 \to \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 5 & -12 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 17 + \frac{7}{3} \end{vmatrix} = E$$

$$|A| = |B| = |E| = 1 \cdot (-1) \cdot 3 \cdot \left(17 + 7/3\right)$$

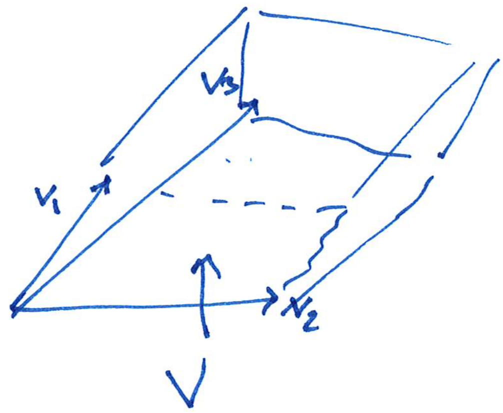
$$= -(51 + 7) = \underline{\underline{-58}}$$

Applications:



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \cancel{A} \pm A$$

$$\underline{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$



$$\begin{array}{c} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \pm V \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{array} \end{array}$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 4 \cdot \underline{v}_1$$

Ex: $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ $\quad |A| = 0$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad |A| = 0$$

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② Linear systems, inverse matrices and determinants

Linear system: $n \times n$ linear system

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \iff \underset{n\times n\text{-matrix}}{A} \cdot \underset{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}{\underline{x}} = \underset{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}{\underline{b}}$$

Fact:

$|A| \neq 0$: one unique solution

$|A| = 0$: either no solutions or infinitely many solutions

Ex:

$$\begin{aligned} x + y + z &= 47 \\ x - y + z &= 14 \\ x + 4y + 16z &= 3 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 4 & 16 \end{vmatrix}$$

$$= 1(-20) - 1(15) + 1(5)$$

$$= -20 - 15 + 5 = \underline{-30} \neq 0$$

$\Rightarrow$ the linear system has a unique solution

$$A\underline{x} = \underline{b} \Rightarrow \cancel{A^{-1}}\cancel{A}\underline{x} = A^{-1}\underline{b}$$

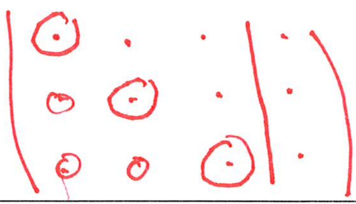
$$\Rightarrow \underline{x} = A^{-1} \cdot \underline{b}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 47 \\ 1 & -1 & 1 & 14 \\ 1 & 4 & 16 & 3 \end{array}\right)$$

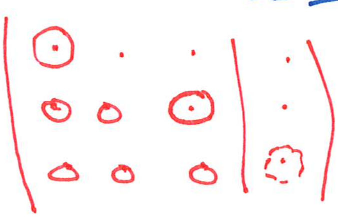
$\downarrow$

$\vdots$

$\downarrow$



$|A| \neq 0$



$|A| = 0$

How to compute inverse matrices using determinants:

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 4 & 16 \end{pmatrix}$

$|A| = -30 \neq 0$

$\Downarrow$

A^{-1} exists

$(A \mid I) \to \cdots \to (B \mid C)$ reduced echelon form

$B = I$: $A^{-1} = C$

$B \neq I$: A^{-1} does not exist

A $n \times n$ matrix: A has an inverse

$\Updownarrow$

the (reduced) echelon form of A has a pivot in every col.

$\Updownarrow$

$|A| \neq 0$

Result:

A $n\times n$-matrix

$|A| = 0 \Rightarrow A^{-1}$ does not exist

$|A| \neq 0 \Rightarrow A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$,

where $\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 4 & 16 \end{pmatrix}$ $|A| = -30$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$C_{11} = -20$ $C_{12} = -15$ $C_{13} = 5$

$C_{21} = -12$ $C_{22} = 15$ $C_{23} = -3$

$C_{31} = 2$ $C_{32} = 0$ $C_{33} = -2$

$$A^{-1} = \frac{1}{-30} \begin{pmatrix} -20 & -15 & 5 \\ -12 & 15 & -3 \\ 2 & 0 & -2 \end{pmatrix}^T$$

$$= \frac{1}{-30} \begin{pmatrix} -20 & -12 & 2 \\ -15 & 15 & 0 \\ 5 & -3 & -2 \end{pmatrix}$$

Ex:

$$x+y+z=47$$
$$x-y+z=14$$
$$x+4y+16z=3$$

$$A=\begin{pmatrix}1&1&1\\1&-1&1\\1&4&16\end{pmatrix} \qquad |A|=-30\neq 0$$

$\Rightarrow$ there is one unique solution

$\underline{A}\,\underline{x}=\underline{b} \quad | A^{-1}\cdot$

$\underline{x}=A^{-1}\cdot\underline{b}$

$$A^{-1}=\frac{1}{-30}\begin{pmatrix}-20&-12&2\\-15&15&0\\5&-3&-2\end{pmatrix}$$

$$\underline{x}=\frac{1}{-30}\begin{pmatrix}-20&-12&2\\-15&15&0\\5&-3&-2\end{pmatrix}\cdot\begin{pmatrix}47\\14\\3\end{pmatrix}=\frac{1}{-30}\begin{pmatrix}-20\cdot47-12\cdot14+2\cdot3\\-15\cdot47+15\cdot14\\5\cdot47-3\cdot14-2\cdot3\end{pmatrix}$$

$$=\frac{1}{-30}\begin{pmatrix}-1102\\-495\\187\end{pmatrix}=\underline{\underline{\begin{pmatrix}1102/30\\495/30\\-187/30\end{pmatrix}}}$$

Transpose of a matrix

$A \rightsquigarrow A^T$ transpose of A

$m\times n$ matrix $\qquad n\times m$ matrix

first row of A = first col. of A^T

$\vdots$

$$A=\begin{pmatrix}1&2\\3&4\end{pmatrix} \qquad A^T=\begin{pmatrix}1&3\\2&4\end{pmatrix}$$

A is called symmetric if $A^T=A$.