Plan

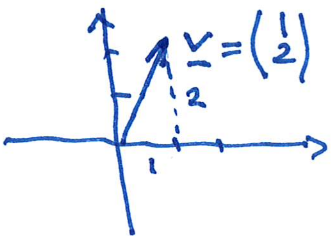
① Vectors and matrices

Vector:

A column vector is a matrix with only one column.

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \vec{w}$

A vector is an entity with a direction and magnitude

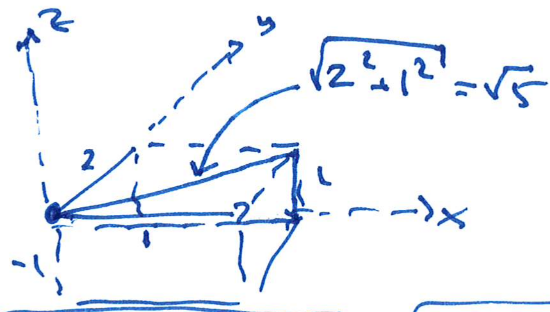


Length of a vector:

$$\|\underline{v}\| = \left\| \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Ex: $\|\underline{v}\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$

$$\|\underline{w}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \underline{\underline{\sqrt{6}}}$$



$$\sqrt{\left(\sqrt{2^2 + 1^2}\right)^2 + 1^2} = \sqrt{2^2 + 1^2 + (-1)^2}$$

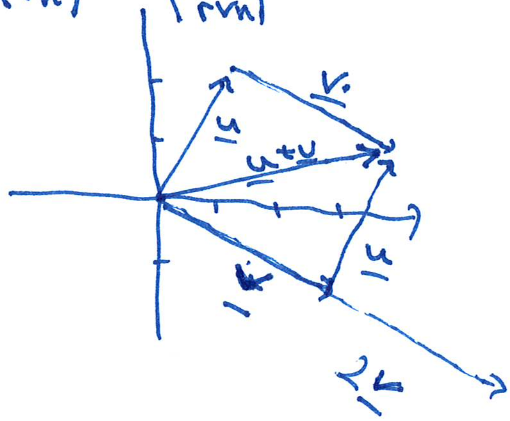
Operations on vectors:

- addition: $\underline{v}+\underline{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix}$

- subtraction: $\underline{v}-\underline{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} - \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1-w_1 \\ v_2-w_2 \\ \vdots \\ v_n-w_n \end{pmatrix}$

- scalar multiplication: $r\cdot \underline{v} = r\cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ \vdots \\ rv_n \end{pmatrix}$

Ex: $\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{=\underline{u}} + \underbrace{\begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{=\underline{v}} = \underline{\underline{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}}$



$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}}$

$2\cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 6 \\ -2 \end{pmatrix}}}$

Matrices:

An $m \times n$ matrix A is a rectangular array of numbers with m rows and n columns.

Ex: $A = \begin{pmatrix} 1 & 7 & 4 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ number in row 1, col. 2

2x3 matrix

Operation on matrices:

- addition / subtraction: $A \pm B$
  (defined when A, B have the same size)

- scalar multiplication: $r \cdot A$

Ex:

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 3 & -2 & 10 \end{pmatrix}$$

$$2 \cdot \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 4 & 2 & 0 \end{pmatrix}$$

Inner product of vectors

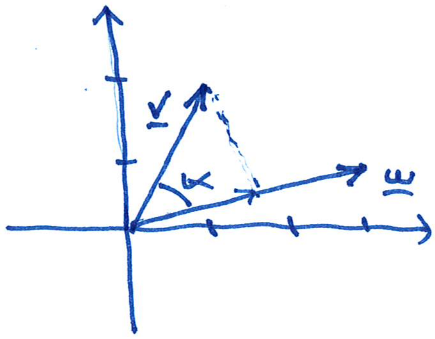
Defn: $\underline{v} = (v_1, v_2, \ldots, v_n)$
$\underline{w} = (w_1, w_2, \ldots, w_n)$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\boxed{\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 \cdot w_2 + \ldots + v_n \cdot w_n}$$

inner product (dot product) of $\underline{v}, \underline{w}$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 1 \cdot 3 + 2 \cdot 1 = \underline{\underline{5}}$



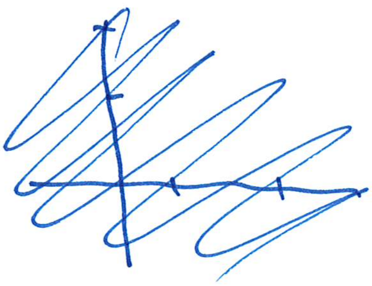
$\|\underline{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$
$\|w\| = \sqrt{3^2 + 1^2} = \sqrt{10}$

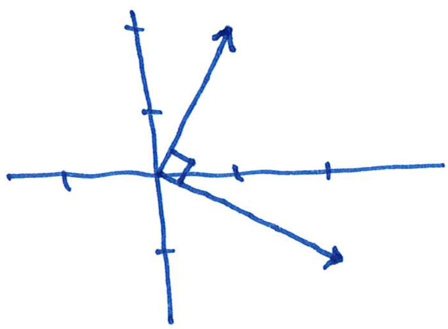
$\|\underline{v}\| \cdot \|\underline{w}\| = \sqrt{5} \cdot \sqrt{10}$
$= \sqrt{50} \simeq 7.1$

$\left(\underline{v} \cdot \underline{w} = \|\underline{v}\| \cdot \|\underline{w}\| \cdot \cos \alpha \right)$

Result: $\underline{v} \cdot \underline{w} = 0 \iff$ angle between $\underline{v}, \underline{w}$ is $90°$
$\underline{v} \perp \underline{w}$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 1 \cdot 2 + 2 \cdot (-1) = 0$





Matrix multiplication:

A ($m \times n$ matrix), $\underline{v}$ (n-vector) $\rightsquigarrow$ $A \cdot \underline{v}$ (m-vector)

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad A \cdot \underline{v} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$

$2 \times \underline{3}$ $\qquad \underline{3}$-vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 1 = 8$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot 1 + (-1) \cdot 2 + 0 \cdot 1 = -1$$

A: $m \times n$-matrix

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

n-vectors $\quad$ m-vectors

$\underline{v} \mapsto A \cdot \underline{v}$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ $\quad$ $1 \cdot 3 + 2 \cdot 0$, $\quad 2 \cdot 3 + (-1) \cdot 0$

$2 \times \underline{2} \qquad \underline{2} \times 1$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

$$\underset{m\times n \text{ matrix}}{A} \cdot \underset{n\times p \text{ matrix}}{B} \leadsto \underset{m\times p \text{ matrix}}{A\cdot B}$$

Ex:

$$A\cdot B = \underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}} = \underline{\underline{\begin{pmatrix} 1 & 9 \\ 5 & 6 \end{pmatrix}}}$$

$$B\cdot A = \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 5 & 7 \\ 7 & 2 \end{pmatrix}}}$$

Note: matrix multiplication is noncommutative

$(A\cdot B \neq B\cdot A)$

Ex:

$$\begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}}}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix

$$\boxed{\begin{array}{l} A\cdot I = A \\ I\cdot A = A \\ \text{for any matrix } A \end{array}}$$

Ex:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

identity matrix

$A \cdot I = I \cdot A$
for any matrix A.

Ex:

$$(A+B)^2 = (A+B)\cdot(A+B) = A\cdot A + A\cdot B + B\cdot A + B\cdot B$$

$$= \underline{\underline{A^2 + AB + BA + B^2}}$$

Connections with linear systems:

$$\left.\begin{aligned} x+y+z+w &= 7 \\ x-y+4w &= 11 \\ 2x+3y+z &= 14 \end{aligned}\right\}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

coefficient matrix

$$\underline{b} = \begin{pmatrix} 7 \\ 11 \\ 14 \end{pmatrix}$$

$$A\cdot\underline{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z+w \\ x-y+4w \\ 2x+3y+z \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \\ 14 \end{pmatrix} = \underline{b}$$

Linear system in matrix form: $\boxed{A\cdot\underline{x} = \underline{b}}$

Augmented matrix:

$$(A \mid \underline{b})$$

③ Inverse matrices:

Defn: Let A be a matrix. An inverse matrix of A is a matrix A^{-1} such that $\left\{\begin{array}{l} A\cdot A^{-1} = I \\ A^{-1}\cdot A = I \end{array}\right\}$.

Note: If A has an inverse A^{-1}, then it is unique.

Assume that B, C are both inverses of A.

$$\left.\begin{array}{l} AB = I \\ BA = I \end{array} \text{ and } \begin{array}{l} AC = I \\ CA = I \end{array}\right\} \Rightarrow \begin{array}{ccc} & BAC & \\ \| & & \| \\ (BA)\cdot C & & B\cdot(AC) \\ \| & & \| \\ I\cdot C & & B\cdot I \\ \| & & \| \\ C & & B \end{array}$$

For numbers:

a : a number

$a\cdot a^{-1} = 1$

$\boxed{a^{-1} = 1/a \text{ if } a \neq 0}$

Ex: $ax = b$

$\cancel{a^{-1}}\cdot \cancel{a}x = a^{-1}b$

$x = \frac{1}{a}b$

$x = \frac{b}{a}$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

solve these eqn's for a, b, c, d:

$$\left\{\begin{array}{l} 2a + b = 1 \\ a + 2b = 0 \\ \vdots \\ b + 2d = 1 \end{array}\right.$$

Methods: for determining whether A has an inverse, and computing A^{-1}.

Alt. I: Gaussian elimination

$$(A \mid I) \to \dots \to (B \mid C)$$

reduced echelon form

Answers:

If $B \neq I$, then A^{-1} does not exist.

If $B = I$, then $A^{-1} = C$.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 3 & -1 & 2 \end{array}\right) \cdot \tfrac{1}{3}$$

$$\to \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right)$$

reduced echelon form

$= I$ $\quad$ $= A^{-1}$

$$A^{-1} = \underline{\underline{\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}}} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$$

Note:

i) If A has an inverse, then it is square $(m = n)$

ii) If A is square $(n \times n)$, then it has an inverse if and only if # pivots in $A = n$.

Alt 2: Using determinants

Case I: $n=2$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Formula:

$$\boxed{\begin{array}{ll} A^{-1} = \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & \text{if } \overbrace{ad-bc}^{\det(A)} \neq 0 \\ A^{-1} \text{ does not exist} & \text{if } ad-bc = 0 \end{array}}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \det(A) = 2\cdot 2 - 1\cdot 1 = 4-1 = \underline{3} \neq 0$

$$A^{-1} = \underline{\underline{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}} = \underline{\underline{\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}}}$$

$|A| = ad-bc$
determinant of A

$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
adjungated matrix of A

Example:

$$2x + y = 14$$
$$x + 2y = 7$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 7 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 21 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 7 \\ 0 \end{pmatrix}}}$$

$$(x, y) = \underline{\underline{(7, 0)}}$$

$$A \cdot \underline{x} = \underline{b} \quad \mid A^{-1} \cdot$$
$$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$$
$$I \underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$