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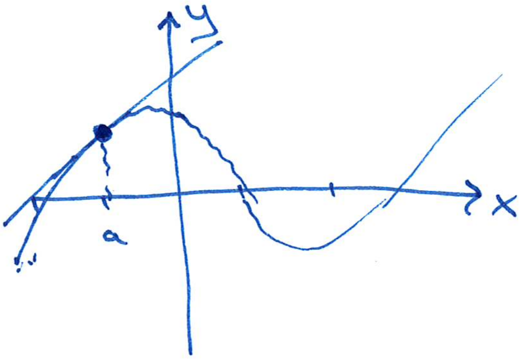
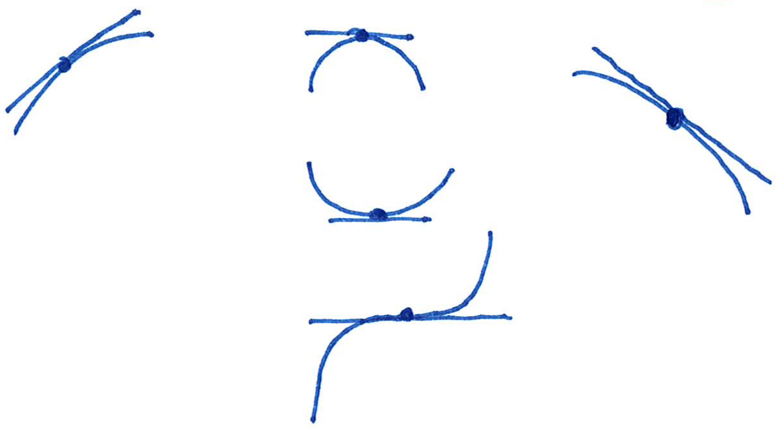
① Functions and derivatives



graph of $y=f(x)$

$f'(a) :=$ slope of the tangent line of $y=f(x)$ at $x=a$

$f'(a)>0$ $f'(a)=0$ $f'(a)<0$

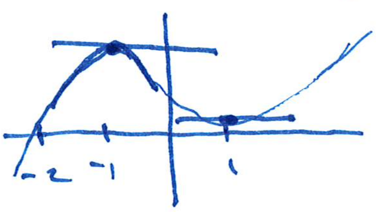


How to compute $f'(a)$:

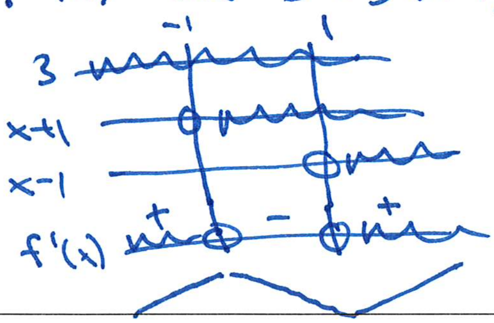
Ex: $f(x)=x^3-3x+2$

$f'(x)=3x^2-3$ ← derivation rules

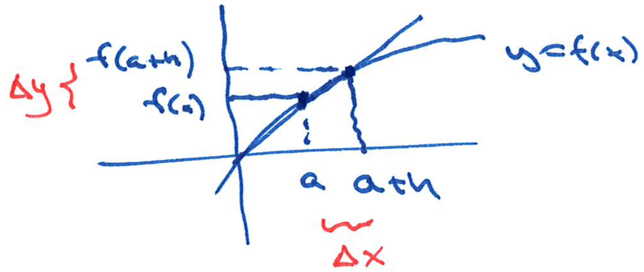
$$f'(-1)=3\cdot(-1)^2-3 = 3-3=0$$



$$f'(x)=3x^2-3=3(x^2-1)=3(x+1)(x-1)$$



From the definition: $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \begin{matrix} \leftarrow \Delta y \\ \leftarrow \Delta x \end{matrix}$$



Using derivation rules:

i) $(x^n)' = n\cdot x^{n-1}$ for any number n $\quad f(x)=x^n \Rightarrow f'(x) = n\cdot x^{n-1}$

ii) $(u\pm v)' = u' \pm v'$ for any expr. u, v

iii) $(c\cdot u)' = c\cdot u'$ for any constant c, exp. u

iv) $(u\cdot v)' = u'\cdot v + u v'$

v) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

} for any expr. u, v

Ex: $$\left(\frac{x}{x-2}\right)' = \frac{1\cdot(x-2) - x\cdot 1}{(x-2)^2} = \underline{\underline{\frac{-2}{(x-2)^2}}}$$

$$\boxed{(ax+b)' = a}$$

vi) Chain rule: $\boxed{[f(u(x))]' = f'(u(x))\cdot u'(x)}$

$$(\sqrt{u})' = (u^{1/2})' = \frac{1}{2}u^{-1/2} = \frac{1}{2}\frac{1}{u^{1/2}} = \frac{1}{2\sqrt{u}}$$

Ex: $f(x) = \sqrt{1-x^2} = \underbrace{\sqrt{u}}_{\text{outer fn.}}, \quad \underbrace{u = 1-x^2}_{\text{kernel, inner fn.}}$

$$f'(x) = \frac{1}{2\sqrt{u}}\cdot(-2x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{u}} = \underline{\underline{\frac{-x}{\sqrt{1-x^2}}}}$$

vii) $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x}$ } base $e = 2.7182\ldots$

$(a^x)' = a^x \cdot \ln(a)$, $(\log_a(x))' = \frac{1}{x \cdot \ln(a)}$ } base $a > 0$, $a \neq 1$

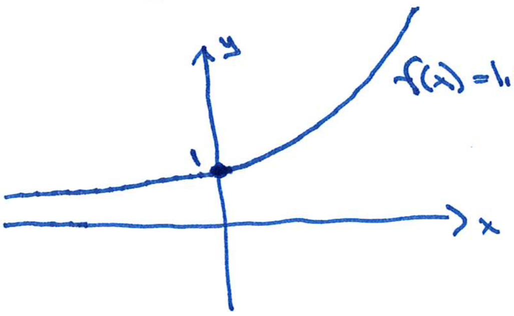
(2) Exponential functions and logarithms

Assume that $a > 0$, $a \neq 1$:

$f(x) = a^x$ exponential function with base a

Ex: $f(x) = 1{,}04^x$

$a = 1.04$



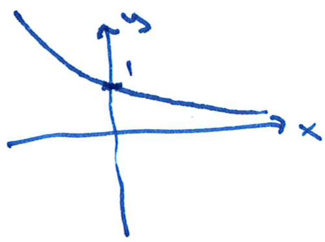
Exponential fn. when $a > 1$:

- f increasing and continuous
- $D_f = \mathbb{R}$, $V_f = (0, \to)$
- when $x \to \infty$, $f(x) \to \infty$; $x \to -\infty$, $f(x) \to 0$
- derivative $(a^x)' = a^x \cdot \ln(a)$

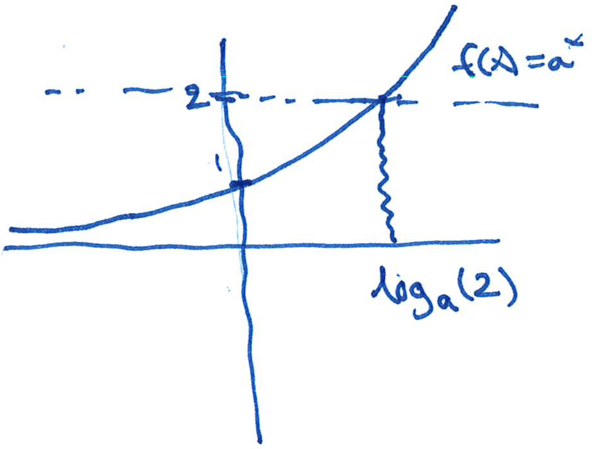
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1{,}04^{x+h} - 1{,}04^x}{h} = \lim_{h \to 0} \frac{1{,}04^x \cdot 1{,}04^h - 1{,}04^x}{h}$$

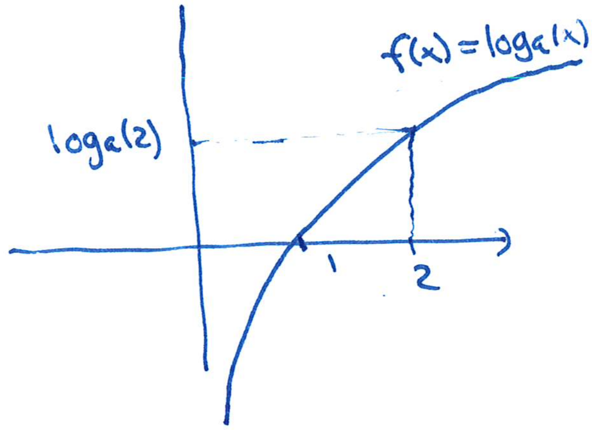
$$= 1{,}04^x \cdot \boxed{\lim_{h \to 0} \frac{1{,}04^h - 1}{h}} = 1{,}04^x \cdot \ln(1{,}04)$$

Ex: $f(x) = 0.96^x$ ($a < 1$)



- f decreasing and cont.
- $D_f = \mathbb{R}$, $V_f = (0, \to)$
- when $x \to \infty$, $f(x) \to 0$; $x \to -\infty$, $f(x) \to \infty$
- derivative: $(a^x)' = a^x \cdot \ln(a)$





Logarithms: $f(x) = \log_a(x)$:= inverse function of a^x

$a>1$: $D_f = (0, \infty)$ $V_f = \mathbb{R}$

f is increasing and continuous

when $x \to \infty$ $f(x) \to \infty$

$x \to 0^+$ $f(x) \to -\infty$

derivative: $(\log_a(x))' = \dfrac{1}{x \cdot \ln(a)}$

defining property: $a^{\log_a(x)} = x$

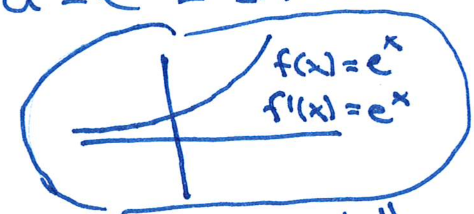
and

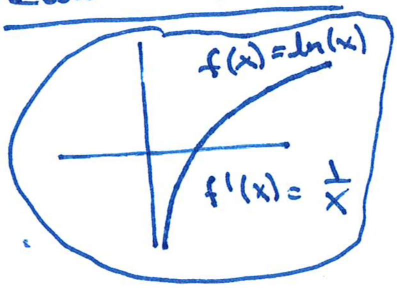
$\log_a(a^x) = x$

Important special case: $a = e = 2.71828\ldots$ **Euler's number**

$f(x) = e^x$

$f(x) = \ln(x) := \log_e(x)$





$\ln(e) = \ln(e^1) = 1$

Derivatives:

$(e^x)' = e^x$ $\qquad (e^u)' = e^u \cdot u'$

$(\ln x)' = \dfrac{1}{x}$ $\qquad \ln(u)' = \dfrac{1}{u} \cdot u'$

Rules for powers:

i) $a^m \cdot a^n = a^{m+n}$

ii) $a^m / a^n = a^{m-n}$

iii) $(a^m)^n = a^{m \cdot n}$

Rules for logarithms:

i) $\ln(x \cdot y) = \ln(x) + \ln(y)$

ii) $\ln(x/y) = \ln(x) - \ln(y)$

iii) $\ln(x^y) = y \cdot \ln(x)$

Ex: $f(x) = 2e^x - 3$

$f'(x) = \underline{2 \cdot e^x}$

Intersection with x-axis:

$f(x) = 2e^x - 3 = 0$

$\frac{2e^x}{2} = \frac{3}{2}$

$e^x = 3/2$

$\ln(e^x) = \ln(3/2)$

$x = \ln(3/2)$

$x = \underline{\underline{\ln 3 - \ln 2}}$

③ Higher derivatives and convexity

Ex: $f(x) = \underline{x^3 - 3x + 2}$

$f'(x) = \underline{3x^2 - 3}$

$f''(x) = 3 \cdot 2x = \underline{6x}$

$f'''(x) = \underline{6}$

$f^{(4)}(x) = \underline{0}$

Defn:

f is convex in an interval I if $f''(x) \geq 0$ for all x in I

f is concave in an interval I if $f''(x) \leq 0$ for all x in I

$f''(x) = 6x$

$f'' = 6x$ ——— 0 ———

f is convex in $[0, \to)$

f is concave in $(\leftarrow, 0]$

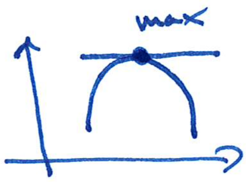
<u>Optimization problems</u> in one variable

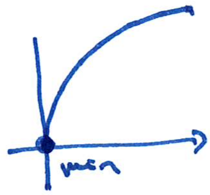
max/min $f(x)$

<u>Definition:</u> A stationary pt of f is a pt. where $f'(x)=0$.

<u>Fact:</u>

$$\boxed{\begin{array}{l}\text{A max/min for } f \text{ is either}\\ \text{i) stationary pt } f'(x)=0\\ \text{ii) a pt where } f'(x) \text{ doesn't exist}\\ \text{iii) a boundary pt}\end{array}}$$

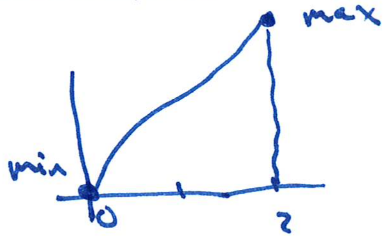




$f(x)=\sqrt{x}$, $x\geq 0$

$f'(x)=\frac{1}{2\sqrt{x}}$

$f'(0)$ doesnot exist



$f(x)$ increasing

$D_f=[0,2]$

<u>Ex:</u> $f(x)=4-x^2$

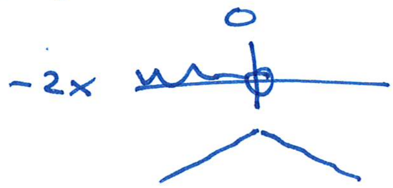
$f'(x)=\underline{-2x}=0$

<u>Stat.pt:</u> $x=0$

$f''(x)=-2$

<u>Methods</u> to classify $x=0$

i) Sign diagram for $f'(x)$:



<u>$x=0$ is max</u>

ii) Second derivative test:

$$\boxed{\begin{array}{l}x=a \text{ is a stationary pt}\\ \text{then:}\\ f''(a)>0 \Rightarrow x=a \text{ local min}\\ f''(a)<0 \Rightarrow x=a \text{ local max}\end{array}}$$

$f''(0)=-2$ $x=0$ is <u>local max</u>

Convex optimization:

f is convex $\Rightarrow$ any stationary pt is global min = min

f is concave $\Rightarrow$ — " — is global max = max

convex → min

concave → max

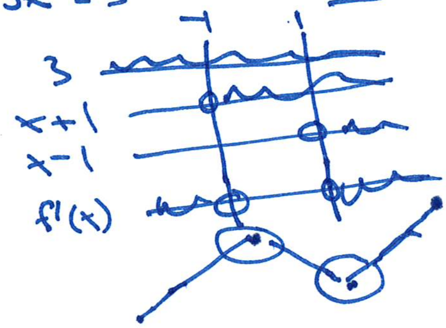
Ex: $f(x) = x^3 - 3x + 2$

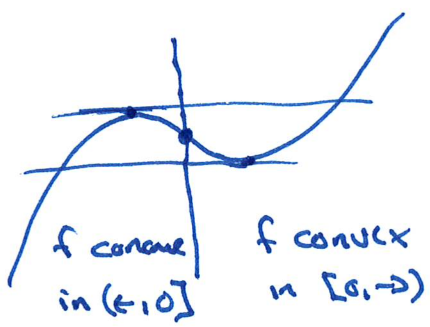
$f'(x) = 3x^2 - 3 = 0 \Rightarrow 3x^2 = 3 \quad x^2 = 1$

$= 3(x^2 - 1)$

$= 3(x+1)(x-1)$

Stationary pts: $x = 1$, $x = -1$





$x = -1$: local max   no max

$x = 1$: local min   no min

Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x), \quad 0 \le x < 1$

$\ln(u)' = \frac{1}{u} \cdot u'$

$$f'(x) = \frac{3}{5} \cdot \frac{1}{1+x} \cdot 1 + \frac{2}{5} \cdot \frac{1}{1-x} \cdot (-1)$$

$$= \frac{3(1-x)}{5(1+x)(1-x)} + \frac{-2(1+x)}{5(1-x)(1+x)}$$

$$= \frac{3(1-x) - 2(1+x)}{5(1+x)(1-x)} = \frac{1-5x}{5(1-x)(1+x)} = 0$$

$$1 - 5x = 0$$

Stationary pt: $x = 1/5$

$$f''(x) = \frac{3}{5}\left[(1+x)^{-1}\right]' - \frac{2}{5}\left[(1-x)^{-1}\right]'$$

$$= \frac{3}{5}(-1)(1+x)^{-2} \cdot 1 - \frac{2}{5} \cdot (-1)(1-x)^{-2} \cdot (-1)$$

$$= -\frac{3}{5}\frac{1}{(1+x)^2} - \frac{2}{5}\frac{1}{(1-x)^2} < 0 \quad \text{for all } x$$

f is concave 

$\Downarrow$

$x = 1/5$ is max

$f(x) = \frac{3}{5}\ln(1+x)$
$+ \frac{3}{5}\ln(1-x)$,
$0 \le x < 1$