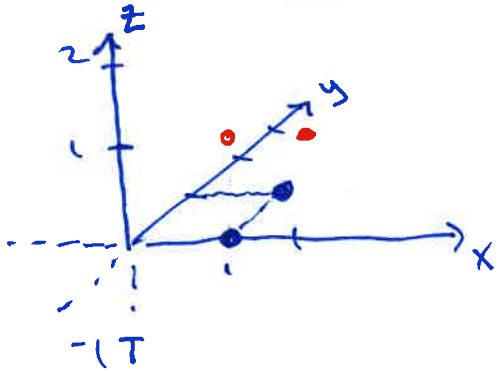
Plan

1. Functions in two variables and partial derivatives
2. Unconstrained optimization
3. Constrained optimization and Lagrange multipliers

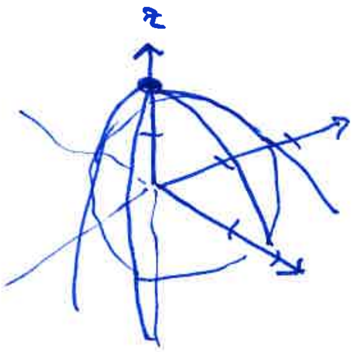
① Function in two variables

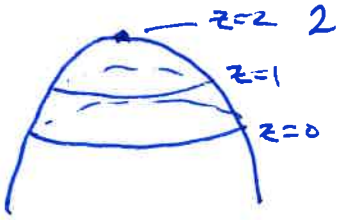
Ex: $f(x,y) = \underbrace{x^3 - xy + y^2}_{\text{functional expression}}$ $f(1,0) = \underline{1}$ $f(1,1) = \underline{1}$

$D_f = \{(x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$ $z = f(x,y)$



$f(x,y) = 2 - x^2 - y^2$ $D_f = \mathbb{R}^2$





$f(x,y) = 0$

$2 - x^2 - y^2 = 0$

$2 = x^2 + y^2$

circle, center $(0,0)$, radius $\sqrt{2}$

Partial derivatives:

Ex: $f(x,y) = x^3 - xy + y^2$

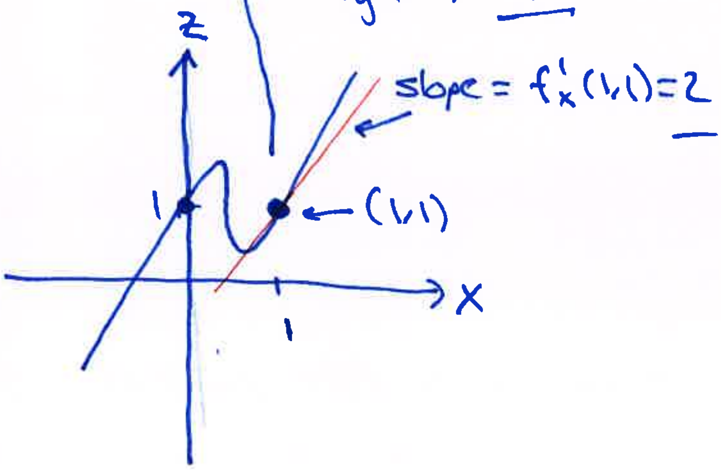
$f'_x(x,y) = 3x^2 - y\cdot 1 + 0 = \underline{3x^2 - y}$

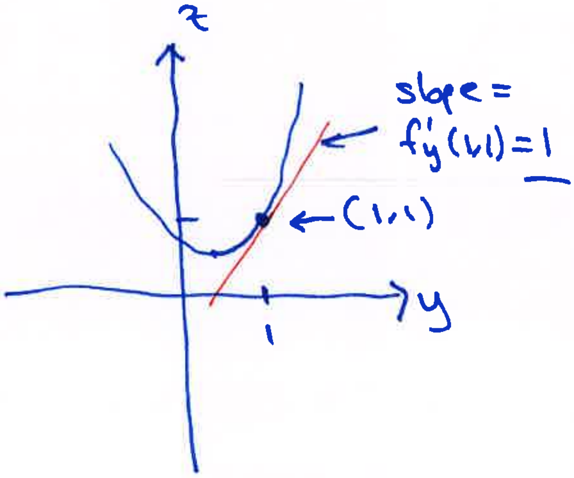
$f'_y(x,y) = 0 - x\cdot 1 + 2y = \underline{-x+2y}$

At $(1,1)$: $f(1,1) = 1$

$f'_x(1,1) = \underline{2}$

$f'_y(1,1) = \underline{1}$





$\underline{y=1}$: $z = f(x,1) = x^3 - x + 1$

$\underline{x=1}$: $z = f(1,y) = 1 - y + y^2$

Hessian matrix:

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = \underline{3x^2 - y}$ $\quad f''_{xx} = \underline{6x}$ $\quad f''_{xy} = \underline{-1}$

$f'_y = \underline{-x+2y}$ $\quad f''_{yx} = \underline{-1}$ $\quad f''_{yy} = \underline{2}$

$$H(f)(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix}$$

Fact: $f''_{xy} = f''_{yx}$ when f is "nice" $\Longleftrightarrow$ $H(f)$ is <u>symmetric</u>

2 Unconstrained optimization

$$\boxed{\max/\min \ f(x,y)}$$

Ex: max/min $f(x,y) = x^3 - xy + y^2$

Stationary pts of f: $\boxed{\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}}$ FOC (first order condition)

Ex: $f = x^3 - xy + y^2$

$f'_x = \boxed{3x^2 - y = 0}$

$f'_y = \boxed{-x + 2y = 0}$

$x = 2y$ (2)

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$3x^2 - y = 0$ (1)

$3(2y)^2 - y = 0$

$12y^2 - y = 0$

$y(12y - 1) = 0$ $\quad$ $y = 0$, $x = 0$ or $y = 1/12$, $x = 1/6$

Stationary pts of f: $(x,y) = (0,0)$, $(1/6, 1/12)$

Candidate pts = pts that could be max/min

i) stationary pts

ii) pts where f'_x og f'_y is not defined

iii) boundary pts

Ex: $(0,0)$, $(1/6, 1/12)$

$f(0,0) = 0$

$f(1/6, 1/12) = \frac{1}{216} - \frac{1}{72} + \frac{1}{144}$

Defn: (x^*, y^*) is max for f if $f(x^*, y^*) \geq f(x,y)$ for all pts (x,y) in D_f.

(x^*, y^*) is local max for f if $f(x^*, y^*) \geq f(x,y)$ for all pts (x,y) close to (x^*, y^*)

Defn: A stationary pt that is not local max or a local min is called a saddle pt.

Second derivative test:

If (x^*, y^*) is a stationary pt for f, and

$$H(f)(x^*, y^*) = \begin{pmatrix} f''_{xx}(x^*, y^*) & f''_{xy}(x^*, y^*) \\ f''_{yx}(x^*, y^*) & f''_{yy}(x^*, y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

then:

$AC - B^2 > 0, \quad A > 0 \Rightarrow (x^*, y^*)$ local min

$AC - B^2 > 0, \quad A < 0 \qquad (x^*, y^*)$ local max

$AC - B^2 < 0 \qquad (x^*, y^*)$ saddle pt.

$AC - B^2 = 0$: no conclusion

$AC - B^2 = \det H(f)(x^*, y^*)$

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y$

$f'_y = -x + 2y$

$H(f) = \begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix}$

Stationary pts:

$(x,y) = \underline{(0,0)}\ ,\ \underline{(1/6, 1/12)}$

$\underline{(0,0)}$: $H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$

$\det = 0 - (-1)^2 = -1 < 0$

$\underline{(0,0) \text{ saddle pt}}$

$\underline{(1/6, 1/12)}$: $H(f)(1/6, 1/12) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$\det = 2 - 1 = 1 > 0$

$A = 1 > 0$

$(1/6, 1/12)$ $\underline{\text{local min}}$

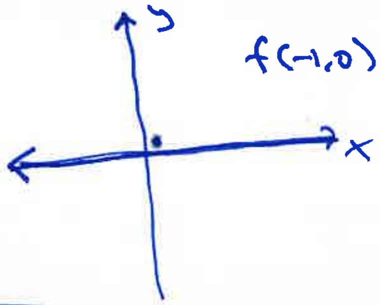
$\underline{\text{no global max}}$

$\underline{\text{cond. for min}}$: $(1/6, 1/12)$ $\underline{\text{not}}$ global min

$f = x^3 - xy + y^2$

$\underline{y=0}$: $f(x,0) = x^3 \to -\infty$ as $x \to -\infty$

$f(-1,0) = -1$



Second derivative test:

$\det H(f)(x^*,y^*) = AC - B^2 > 0$

$AC > B^2 \geq 0$

$\underline{A > 0,\ C > 0}$ local min

$\underline{A < 0,\ C < 0}$ local max

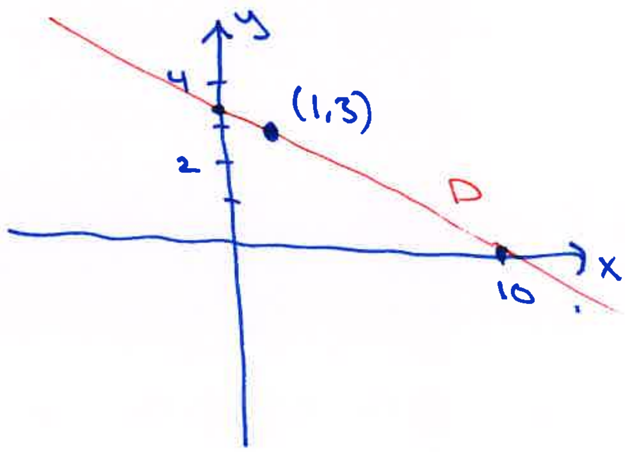
$A = f''_{xx}(x^*,y^*) > 0$

$C = f''_{yy}(x^*,y^*) > 0$

3 Constrained optimization and Lagrange multipliers

max/min $f(x,y)$ when $g(x,y) = a$

Ex: max/min $\underbrace{f(x,y) = x^2+y^2}_{\text{objective fn.}}$ when $\underbrace{\overset{g(x,y)\,=\,a}{x+3y=10}}_{\text{equality constraint}}$



Method of Lagrange multipliers:

$$L(x,y;\lambda) = f(x,y) - \lambda\cdot(g(x,y)-a)$$
$$= x^2+y^2-\lambda(x+3y-10)$$

$$\boxed{\begin{aligned} L'_x &= 2x-\lambda(1)=0 \\ L'_y &= 2y-\lambda(3)=0 \\ L'_\lambda &= -1\cdot(x+3y-10)=0 \end{aligned}}$$

Lagrange conditions = stationary pts of L

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Solutions $(x,y;\lambda)$ = candidates for max/min

D: the set of admissible pts

$$\begin{aligned} L'_x &= 2x-\lambda=0 \\ L'_y &= 2y-3\lambda=0 \end{aligned} \Bigg\}\ \text{FOC}$$
$$x+3y=10 \ \}\ \text{C}$$

(1) $\lambda = 2x$:

(2) $2y-3\cdot 2x=0$

$2y=6x$ $\quad y=3x$

(3) $x+3\cdot 3x=10$

$10x=10$

$x=1 \quad y=3 \quad \lambda=2$

Candidates: $(x,y;\lambda) = (1,3;2)$

$f = f(1,3) = 10$

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Thm

If (x^*, y^*) is a max or min in a Lagrange problem, and (x^*, y^*) satisfies NDCQ (non-degenerate constraint qualification), then there exists a Lagrange multiplier λ such that $(x^*, y^*; \lambda)$ satisfy FOC+C.

(x^*, y^*) max / min $\Longrightarrow$ (if NDCQ ok) $(x^*, y^*; \lambda)$ is one of the solutions of FOC+C

NDCQ:

$g(x,y) = a$

$\boxed{x + 3y = 10}$

$g'_x = g'_y = 0$ is not satisfied at any adm. pt.

$g = x + 3y$

$g'_x = 1 \neq 0$

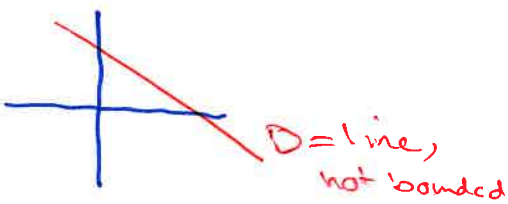
$g'_y = 3 \neq 0$

NDCQ ok in this case

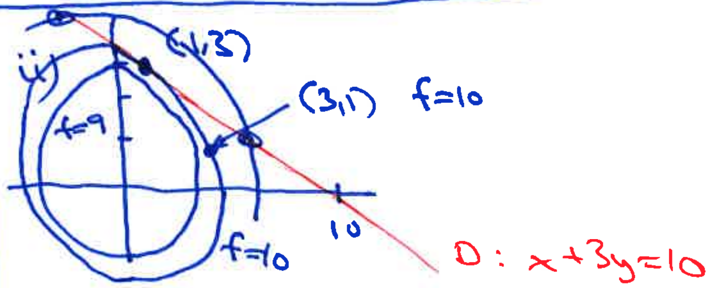
How to find out if a candidate pt is max/min?

i) **Extreme Value Thm:**

If $f(x,y)$ is continuous defined on a closed and bounded set, then f has a max and a min



ii)



Level curve: $f(x,y) = 10$

$x^2 + y^2 = 10$

$f(x,y) = c < 10$

$x^2 + y^2 = c$

$f(x,y) = c > 10$

Concl: $f(x,y)=10$ on $x+3y=10$: one pt $(1,3)$
$f(x,y)<10$ — " — : no pts
$f(x,y)>10$ — " — : two pts

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$f_{min} = \underline{\underline{10}}$ at $(x,y)=(1,3)$ with $\lambda = 2$
there is no max

$\boxed{\text{Solutions of FOC+C}}$ $\Longleftrightarrow$ pts where $\underline{g(x,y)=a}$ meets $f(x,y)=c$ at a tangent