Plan

1 Determinants

2 Linear systems, inverse matrices and determinants

① Determinants

$A \rightsquigarrow \det(A) = |A|$

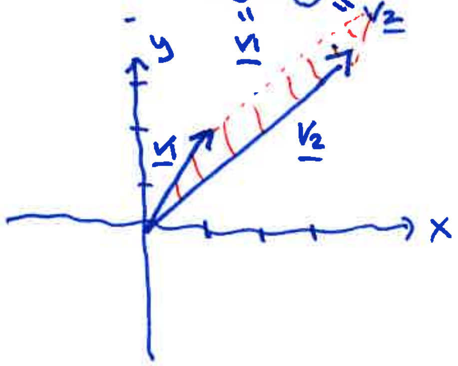
$n \times n$ matrix (square) — determinant of A (a number)

n=2:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex: $\left| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right| = 1 \cdot 4 - 3 \cdot 2 = \underline{-2}$

$\underline{v_1}$, $\underline{v_2}$



± area of the parallellogram spanned by the column vectors of A

$\left| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right| = 3 \cdot 2 - 1 \cdot 4 = \underline{2}$

$\underline{v_2}$, $\underline{v_1}$

$n > 2$:

Ex: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $|A| = aei + bfg + cdh - ceg - bdi - afh$

Alt I: Works for $n=2$ and $n=3$ but not for $n \geq 4$!

$$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right|$$

Alt 2: Cofactor expansion – works in general

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

(1,1): row 1, column 1

Cofactor expansion along the first row:

$$|A| = a \cdot C_{11} + b \cdot C_{12} + c \cdot C_{13}$$

$$= a \cdot (+1) \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \cdot (-1) \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot (+1) \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= +a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

Defn. of cofactors:

$$C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{M_{ij}}_{\text{minor}}$$

(row i, col j)

Signs:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minors:

M_{ij} = the determinant of the submatrix you get if you delete row i, col j

Ex:

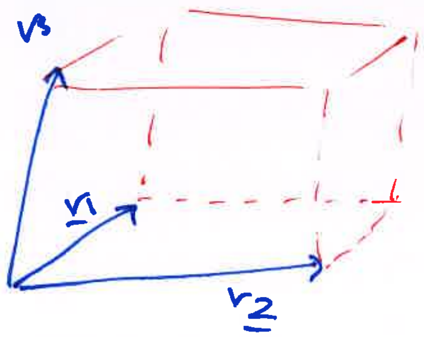
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = +1\cdot(2\cdot 9-4\cdot 3) - 1\,(1\cdot 9 - 1\cdot 4) + 1\,(1\cdot 3 - 1\cdot 2)$$

$$= +6 - 5 + 1 = \underline{\underline{2}}$$

Fact: Cofactor expansion along any row or column give the same result, $|A|$.

$$\begin{array}{c} \begin{array}{ccc} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{array} \\ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \end{array} = +1(3-2) \quad -4(3-1) \quad +9(2-1)$$

$$= +1 - 4\cdot 2 + 9 = \underline{\underline{2}}$$



determinant of $(\underline{v_1}\,|\,\underline{v_2}\,|\,\underline{v_3})$

$=\pm$ volume of parallellepiped spanned by $\underline{v_1}, \underline{v_2}, \underline{v_3}$

Fact: If two columns (rows) in the matrix A are equal (scalar multiple), then $|A|=0$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 2 \\ 3 & -1 & 3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & 4 \\ 3 & 6 & 6 \end{vmatrix} = 0$$

Ex:

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -2 & -2 & 4 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -2 & -2 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= +\big(+4(1+1)\big) - \big(+1 \cdot (1+1)\big)$$

$$= +8 - 2 = \underline{\underline{6}}$$

$$\begin{vmatrix} 7 & 34 & 23 & 7\sqrt{2} \\ 0 & -1 & 14 & 73 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = +7 \cdot \begin{vmatrix} -1 & 14 & 73 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 7\left(-1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}\right) = 7 \cdot (-1) \cdot (1 \cdot 2)$$

$$= \underline{\underline{-14}}$$

Fact: (1) If the matrix A is upper triangular (all entries under the diagonal are zero), then $|A|$ = product of diagonal entries.

(2) Any matrix in echelon form are upper triangular.

Ex:

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 7 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|E| = 1 \cdot 2 \cdot 5 = 10 \qquad |E| = 1 \cdot 0 \cdot 0 = 0$$

Alt 3: Computing determinants using Gauss

Ex:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \underline{\underline{2}}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \\ ]-1 \\ ]-1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \begin{matrix} \\ \\ ]-2 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = E$$

$|A| = \underline{\underline{2}}$

$|E| = 1 \cdot 1 \cdot 2 = \underline{\underline{2}}$

Fact: Let $A \to B$ be an elementary row operation

i) Switch two rows: $|B| = -|A|$

ii) Multiply a row by $c \neq 0$: $|B| = c \cdot |A|$

iii) Add a multiple of one row to another: $|B| = |A|$

Ex:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 0 & 3 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{vmatrix} \begin{matrix} \\ ]-4 \\ ]-3 \\ \\ \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -12 & -13 \\ 0 & -2 & -8 & -10 \\ 0 & 1 & 0 & 2 \end{vmatrix} \begin{matrix} \\ \\ ]8 \\ \\ \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -12 & 3 \\ 0 & -2 & -8 & -10 \\ 0 & 1 & 0 & 2 \end{vmatrix} \begin{matrix} \\ \\ ]2 \\ ]-1 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -12 & 3 \\ 0 & 0 & \boxed{-32} & \boxed{-4} \\ 0 & 0 & \boxed{12} & \boxed{-1} \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & -12 & 3 \\ 0 & -32 & -4 \\ 0 & 12 & -1 \end{vmatrix} = 1 \cdot 1 \cdot (32 + 48) = \underline{\underline{80}}$$

(2) Linear systems, inverse matrices and determinants

Linear system: $(n \times n)$

$$n\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad\quad \vdots \\ \underbrace{a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n}_{n} = b_n \end{cases}$$

Matrix form:

$$A \cdot \underline{x} = \underline{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

coeff. matrix, $n \times n$

Fact:

$|A| \neq 0 \iff A\underline{x} = \underline{b}$ has a unique solution

$|A| = 0 \iff A\underline{x} = \underline{b}$ has either no solutions or inf. many solutions

$$(A \mid \underline{b}) \xrightarrow{\text{elem. row op.}} (E \mid \underline{c})$$

augmented matrix → echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$|A| \neq 0$ one solution

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$|A| = 0$ inf. many sol'n (z free)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 7 \end{array}\right)$$

$|A| = 0$ no solution

Inverse matrices:

A : $n\times n$ matrix

Fact: $\boxed{A \text{ is invertible } \Longleftrightarrow |A| \neq 0 \quad (A^{-1} \text{ exists})}$

Assume A invertible:

$|A| \neq 0$

$A\underline{x} = \underline{b}$: linear system with coeff. matrix A

$$A \cdot \underline{x} = \underline{b}$$

$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b}.$$

$\Uparrow$

one solution

$n=2$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad-bc \neq 0$$

A^{-1} does not exist , $ad-bc=0$

In general:

A $n\times n$-matrix :

$$A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A)$$

$$= \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T, \quad |A| \neq 0$$

A^{-1} does not exist , $|A|=0$

Transpose of a matrix

$$A \rightsquigarrow A^T$$

$m \times n$ matrix $\qquad$ $n \times m$ matrix

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ 2×3 $\qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ 3×2

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 2 & -1 & 3 \end{pmatrix}$ 3×3 $\qquad A^T = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 7 & -1 \\ 3 & 1 & 3 \end{pmatrix}$ 3×3

A is called symmetric if $A^T = A$

$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 7 \end{pmatrix}$ symmetric

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$$|A| = -1(9-3) + 2(9-1) - 4(3-1)$$
$$= -6 + 16 - 8 = \underline{2} \neq 0$$

A <u>invertible</u>

$$A^{-1} = \frac{1}{|A|} \cdot \left(c_{ij} \right)^T = \frac{1}{2} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & +1 \end{pmatrix}^T$$

$$= \underline{\underline{\frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}} = \underline{\underline{\begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}}}$$

$C_{11} = +6$ $C_{12} = -5$ $C_{13} = +1$

$C_{21} = -6$ $C_{22} = +8$ $C_{23} = -2$

$C_{31} = +2$ $C_{32} = -3$ $C_{33} = +1$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $|A| = -\underline{1} \cdot (9-3) + \underline{2}(9-1) - \underline{4}(3-1)$