Plan

1 Optimization problems
2 Integration

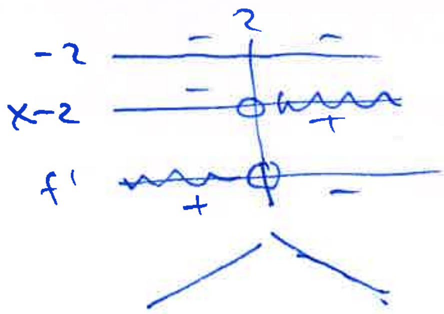
1 Optimization problems

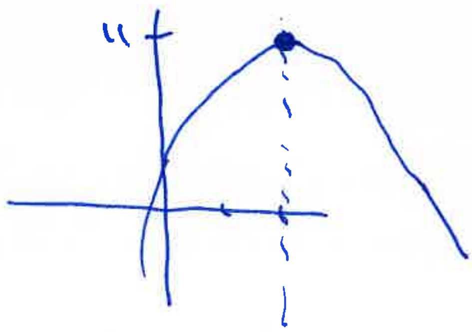
Ex: max/min $f(x) = -x^2 + 4x + 7$

$f'(x) = -2x + 4 = 0$ a stationary pt : $f'(x) = 0$

$x = 2$

$f' = -2(x-2)$





$x = 2$

$f_{max} = 11$

$x = 2$ is a global max

$f''(x) = -2 < 0$ for all x

f is concave 

$x = 2$ is global max

Ex: max/min $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = 0$

$3x^2 - 3 = 0$

$x^2 = 1$

$\underline{x = \pm 1}$

$f''(x) = 6x$

f is neither convex nor concave on $D_f = \mathbb{R}$

Second derivative test:

$f''(-1) = -6 < 0 \Rightarrow x=-1$ local max

$f''(1) = 6 > 0 \qquad x=1$ local min

Ex: max/min $f(x) = \underline{xe^{-x}}$ when $-2 \leq x \leq 3$

Facts:

(1) Extreme Value Thm:

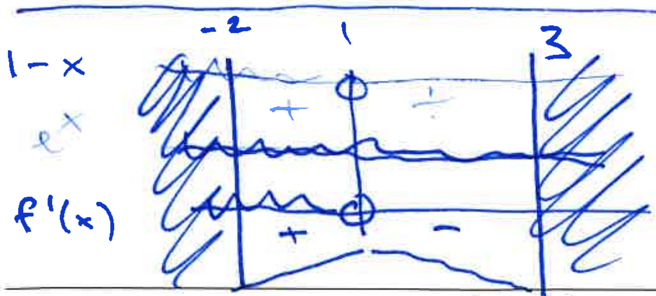
If f is cont. fn. on a closed, bounded interval, then f has a global max and min.

(2) If $x=a$ is a max/min, then either:

i) $x=a$ is a stationary pt.

ii) $f'(a)$ does not exist

iii) $x=a$ is a boundary pt



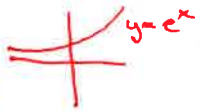
EVT $\Rightarrow$ there is a max/min

$f'(x) = 1 \cdot e^{-x} + x \cdot e^{-x} \cdot (-1)$

$= e^{-x} - xe^{-x}$

$= (1-x)e^{-x} = \dfrac{1-x}{e^x}$

$f'(x) = 0: \quad \dfrac{1-x}{e^x} = 0 \qquad \underline{x=1}$



$\underline{x=-2}$

$\underline{x=3}$

$D_f = [-2, 3]$

$f(1) = 1 \cdot e^{-1} = \frac{1}{e}$

$f(-2) = -2 \cdot e^2$

$f(3) = 3 \cdot e^{-3} = \frac{3}{e^3}$

$f_{min} = \underline{\underline{-2e^2}}$ at $\underline{x=-2}$

$f_{max} = \underline{\underline{1/e}}$ at $\underline{x=1}$

Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$, $0 \le x < 1$

$= 0.6\ln(1+x) + 0.4\ln(1-x)$ $\quad D_f = [0,1)$

$$\boxed{\begin{aligned}(\ln(u(x)))' &= \frac{1}{u(x)} \cdot u'(x) \\ &= \frac{u'(x)}{u(x)}\end{aligned}}$$

$$f'(x) = \frac{3}{5} \cdot \frac{1}{1+x} + \frac{2}{5} \cdot \frac{-1}{1-x}$$

$$= \frac{3/5\,(1-x)}{(1+x)(1-x)} \quad \frac{2/5\,(1+x)}{(1-x)(1+x)}$$

$$= \frac{3/5\,(1-x) - 2/5\,(1+x)}{(1+x)(1-x)} \begin{matrix}\cdot 5 \\ \cdot 5\end{matrix}$$

$$= \frac{3-3x-2-2x}{5(1+x)(1-x)} = \frac{1-5x}{5(1+x)(1-x)} = 0$$

$1-5x=0$

$\underline{x = 1/5}$

(in D_f)

Stationary pts:

$x^* = 1/5 = 0.2$

$$f''(x) = \left(\frac{3/5}{1+x} - \frac{2/5}{1-x}\right)' =$$

$$= \frac{3}{5}\left(-\frac{1}{(1+x)^2}\right) - \frac{2}{5}\left(-\frac{-1}{(1-x)^2}\right)$$

$$= -\frac{3}{5} \cdot \frac{1}{(1+x)^2} - \frac{2}{5}\,\frac{1}{(1-x)^2} < 0$$

for all x

$$\boxed{\begin{aligned}\left(\frac{1}{u(x)}\right)' &= (u(x)^{-1})' \\ &= -1 \cdot u(x)^{-2} \cdot u'(x) \\ &= -\frac{u'(x)}{u(x)^2}\end{aligned}}$$

$\Downarrow$

f concave $\Rightarrow$ $x^* = 1/5 = 0.2$

is global max.

② Integration:

Defn: Let $f(x)$ be any function.
An antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$

Ex: $f(x) = 2x$

$F(x) = x^2$ is an antiderivative
$F(x) = x^2 + 1$ — " —
$F(x) = x^2 + C$ is the general antiderivative

Indefinite integrals:

$$\int \underbrace{f(x)}\,\underbrace{dx} = F(x) + C$$

integration symbol

integration variable is x

the fn. we are supposed to find the antiderivative of

Ex:

$$\int 2x\,dx = \underline{\underline{x^2 + C}}$$

Integration rules:

i) Power rule: $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + C \quad (n \neq -1)$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int u \pm v\,dx = \int u\,dx \pm \int v\,dx$

$\int c \cdot u\,dx = c \cdot \int u\,dx$

(u, v: expr. in x; c: const)

iv) $\int e^x\,dx = e^x + C$

Ex: $\int x^2-3x+2\,dx = \frac{1}{3}x^3 - 3\cdot\frac{1}{2}x^2 + 2x + C$

$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

Ex: $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{3/2}x^{3/2} + C$

$= \frac{2}{3}x\sqrt{x} + C$

$\int \frac{1-x}{x^3}\,dx = \int \frac{1}{x^3} - \frac{x}{x^3}\,dx$

$= \int x^{-3} - x^{-2}\,dx = \frac{1}{-2}x^{-2} - \frac{1}{-1}x^{-1} + C$

$= -\frac{1}{2x^2} + \frac{1}{x} + C$

Ex: $\int \frac{x^3}{1-x}\,dx$ $\quad \int xe^x\,dx$ $\quad \int \ln x\,dx$

$\int x\cdot\sqrt{1-x^2}\,dx$

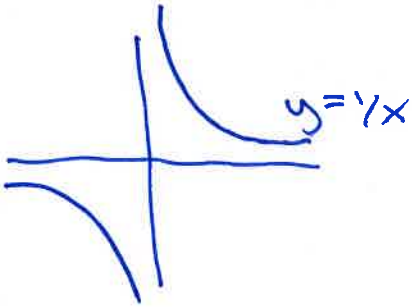
Need to know:

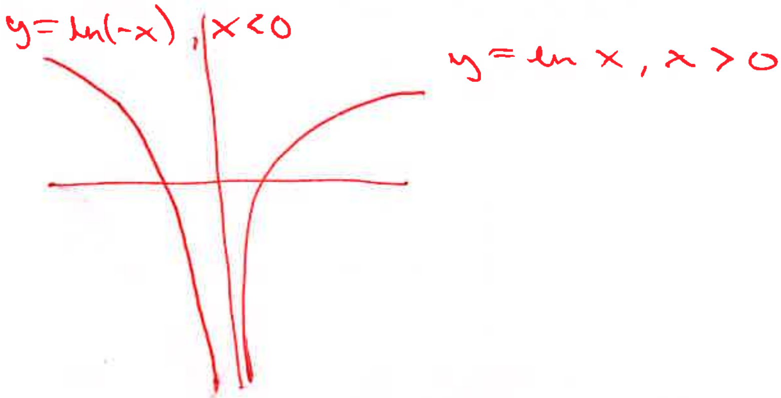
i) Integration by parts

ii) Substitution

iii) Partial fractions

$$\int \frac{1}{x}\,dx = \ln|x| + C = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \end{cases}$$





$$\ln(-x)' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

(a) <u>Integration by parts</u>

$$\int u'\cdot v\,dx = u\cdot v - \int u\cdot v'\,dx$$

$$(uv)' = u'v + uv'$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\Downarrow$$

$$\int u'v\,dx = uv - \int uv'\,dx$$

<u>Ex:</u> $\int x e^x\,dx =$

$$\boxed{\begin{array}{ll} u = x^2/2 & v = e^x \\ u' = x & v' = e^x \end{array}}$$

$$= \frac{x^2}{2}\cdot e^x - \int \frac{x^2}{2}\cdot e^x\,dx$$

$$\int x e^x\,dx =$$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= e^x\cdot x - \int e^x\cdot 1\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= \underline{\underline{xe^x - e^x + C}}$$

Ex: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$$

$$\cancel{= x \cdot \ln x - \frac{x^2}{2} \cdot \ln x + C}$$

$$= x \cdot \ln x - \int 1 \, dx$$

$$= x \cdot \ln x - x + C$$

b) Substitution:

Ex: $\int \frac{1}{1-x} \, dx = \int \frac{1}{u} \left(\frac{1}{-1} du \right)$

$$\boxed{du = u' \, dx}$$

$$\boxed{\begin{array}{l} u = 1 - x \\ du = -1 \, dx \end{array}} \Rightarrow dx = \frac{1}{-1} \cdot du$$

$$= \int -\frac{1}{u} \, du = -1 \cdot \ln |u| + C = -\ln |1-x| + C$$

$$\int x \sqrt{x^2+1} \, dx = \int \cancel{x} \cdot \sqrt{u} \, \frac{1}{2\cancel{x}} \cdot du = \int \frac{1}{2} \sqrt{u} \, du$$

$$\boxed{\begin{array}{l} u = x^2 + 1 \\ du = 2x \, dx \end{array}}$$

$$dx = \frac{1}{2x} du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C = \frac{1}{3} u \sqrt{u} + C = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + C$$

Ex: $\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{u}\cdot 2\sqrt{x}\,du$

$$\boxed{\begin{aligned} u &= 1+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = 2\sqrt{x}\,du$$
$$\sqrt{x} = u-1$$

$$= \int \frac{1}{u}\cdot 2(u-1)\,du = 2\int \frac{u-1}{u}\,du = 2\int 1 - \frac{1}{u}\,du$$

$$= 2\left(u - \ln|u|\right) + C$$

$$= 2\left(1+\sqrt{x} - \ln|1+\sqrt{x}|\right) + C$$

$$= 2 + 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

$$= \underline{2\sqrt{x} - 2\ln(1+\sqrt{x}) + C}$$

c) Partial fractions:

$$\cancel{\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}} \quad \cancel{x^2 = 1-u \quad x = \pm\sqrt{1-u}}$$

$$\cancel{= \int \frac{4}{u}\cdot\frac{1}{-2x}\,du}$$

Ex: $\int \frac{4}{1-x^2}\,dx$

$$\frac{4}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \qquad A=2,\; B=2$$

A, B constants

$$1-x^2 = (1-x)(1+x)$$

$$\frac{4}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \quad \Big|\cdot(1-x)(1+x)$$

$$4 = A(1+x) + B(1-x)$$
$$4 = A + Ax + B - Bx$$
$$4 = (A+B) + (A-B)x$$

$$A-B = 0 \qquad A = B$$
$$A+B = 4 \qquad 2A = 4$$
$$A = 2$$
$$B = 2$$

$$\int \frac{4}{1-x^2}\,dx = \int \frac{2}{1-x} + \frac{2}{1+x}\,dx = -2\ln|1-x| + 2\ln|1+x| + C$$

$$\int \frac{2}{1-x}\,dx = \int \frac{2}{u}\cdot\frac{1}{-1}\,du \qquad \boxed{\begin{matrix} u = 1-x \\ du = -1\cdot dx \end{matrix}}$$

$$= -2\cdot\ln|u| + C$$

$$= -2\ln|1-x| + C$$

$$\int \frac{2}{1+x}\,dx = \int \frac{2}{u}\cdot du \qquad \boxed{\begin{matrix} u = 1+x \\ du = 1\cdot dx \end{matrix}}$$

$$= 2\ln|u| + C = 2\ln|1+x| + C$$

$$= 2\left(\ln|1+x| - \ln|1-x|\right) + C$$

$$= 2\ln\frac{|1+x|}{|1-x|} + C$$

$\ln a - \ln b = \ln a/b$

Ex: $\displaystyle\int \frac{x^2}{1-x^2}\,dx = \int -1 + \frac{1}{1-x^2}\,dx = -x + \int \frac{1}{1-x^2}\,dx$

$$= -x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

Polynomial division:

when deg numerator $\geq$ deg denominator

$$x^2 : (-x^2+1) = -1$$
$$\underline{-(x^2-1)}$$
$$1$$

$$\boxed{\frac{x^2}{1-x^2} = -1 + \frac{1}{1-x^2}}$$