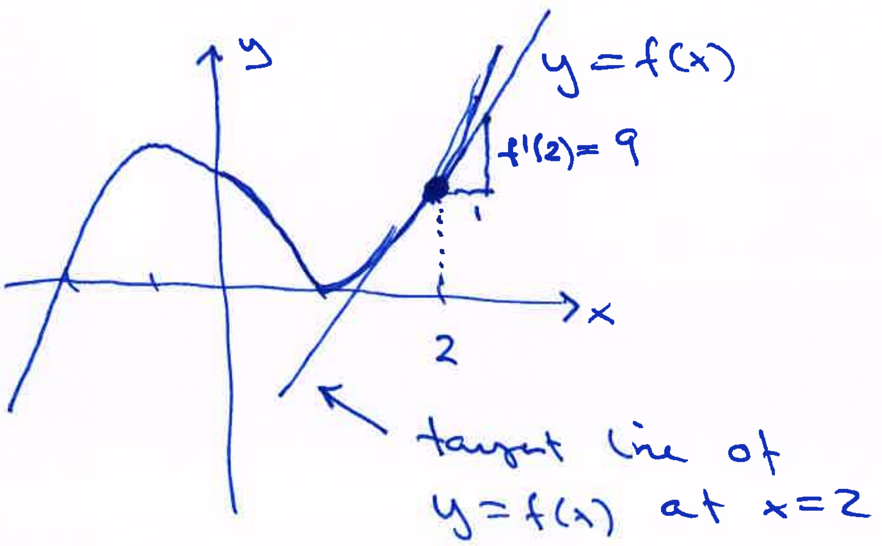
Plan

1. Functions and derivatives
2. Exponential functions and logarithms
3. Higher derivatives and convexity

(i) Functions and derivatives



Derivative of f at $x=2$:

$f'(2) =$ the slope of the tangent line of $y=f(x)$ at $x=2$

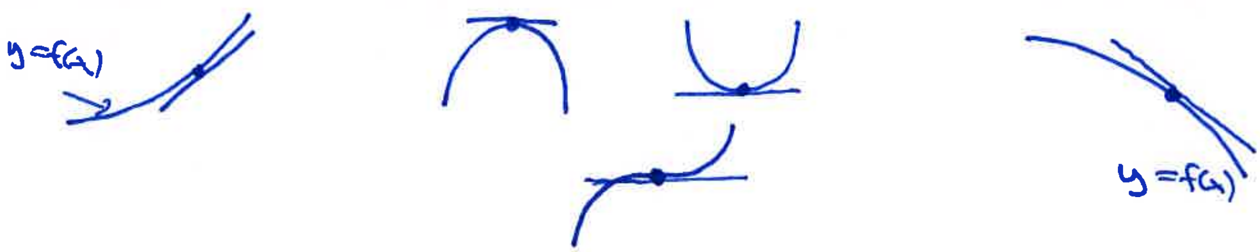
$$f(x)= x^3 - 3x + 2$$

$$f'(x)= \underline{3x^2 - 3} \quad \text{derivative function}$$

$$f'(2) = 3\cdot 2^2 - 3 = 9$$

Interpretation:

$f'(a) > 0$ $\qquad$ $f'(a) = 0$ $\qquad$ $f'(a) < 0$



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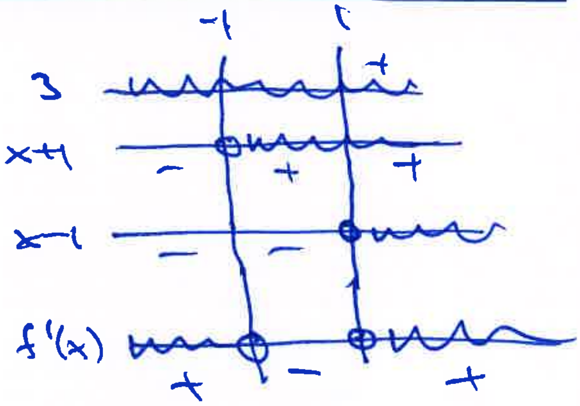
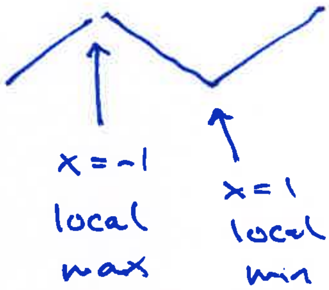
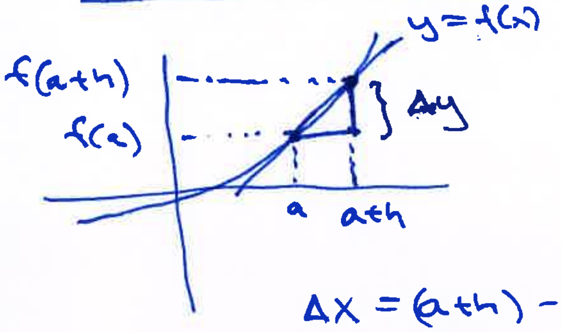
$$f(x) = x^3 - 3x + 2$$
$$f'(x) = 3x^2 - 3$$
$$= 3(x^2-1) = 3(x+1)(x-1)$$





How to compute $f'(x)$:



$$\Delta x = (a+h) - a = h \qquad \Delta y = f(a+h) - f(a)$$

Slope (of secant) $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h)-f(a)}{h}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Derivation rules:

① $(x^n)' = n \cdot x^{n-1}$ (power rule)

② $(u \pm v)' = u' \pm v'$

$(c \cdot u)' = c \cdot u'$

(u, v: expressions in x; c: constant)

③ $(u \cdot v)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$

Ex:
$f(x) = x^2$
$f'(x) = 2x$

Ex:
$f(x) = x^3 + x^2$
$f'(x) = 3x^2 + 2x$

Ex:
$f(x) = 3x^2$
$f'(x) = 3 \cdot 2x = \underline{6x}$

4. Chain rule:

$$f(x) = h(g(x)) = h(u), \quad u = g(x)$$

$$\Downarrow$$

$$f'(x) = h'(u) \cdot g'(x)$$
$$= h'(g(x)) \cdot g'(x)$$

Ex: $f(x) = \sqrt{1-x^2} = \underbrace{\sqrt{u}}_{\text{outer function}} = u^{1/2}, \quad u = \underbrace{1-x^2}_{\text{inner function, kernel}}$

$$f'(x) = \underbrace{\tfrac{1}{2} \cdot u^{-1/2}}_{\text{derivative of the outer function}} \cdot \underbrace{(-2x)}_{\text{derivative of the kernel}}$$

$$= \cancel{\tfrac{1}{2}} \cdot \frac{1}{u^{1/2}} \cdot (-\cancel{2}x) = \frac{-x}{\sqrt{u}} = -\frac{x}{\sqrt{1-x^2}}$$

5. $(a^x)' = a^x \cdot \ln(a)$ $\quad (a > 0)$

(*) $\boxed{(e^x)' = e^x}$ $\quad e = 2.71828\ldots$, (Euler's number)

$(\log_a(x))' = \frac{1}{x} \cdot \frac{1}{\ln(a)}$

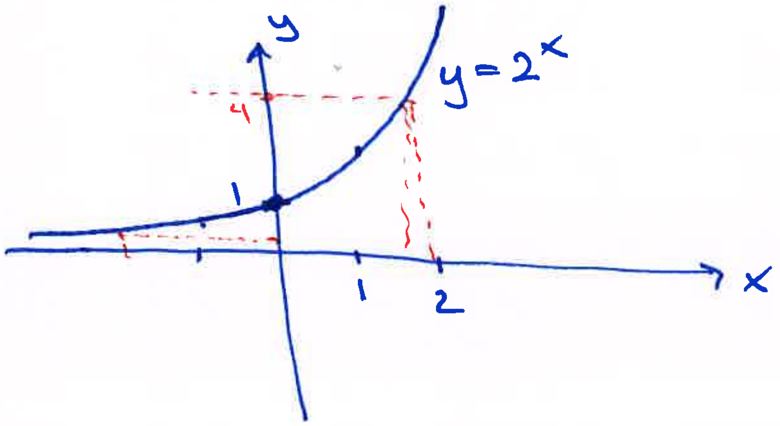
(*) $\boxed{(\ln(x))' = \frac{1}{x}}$

② Exponential functions and logarithms

Base: $a > 0 \quad (a \neq 1)$

Exponential function: $f(x) = a^x$

Ex: $a = 2 \quad f(x) = 2^x$

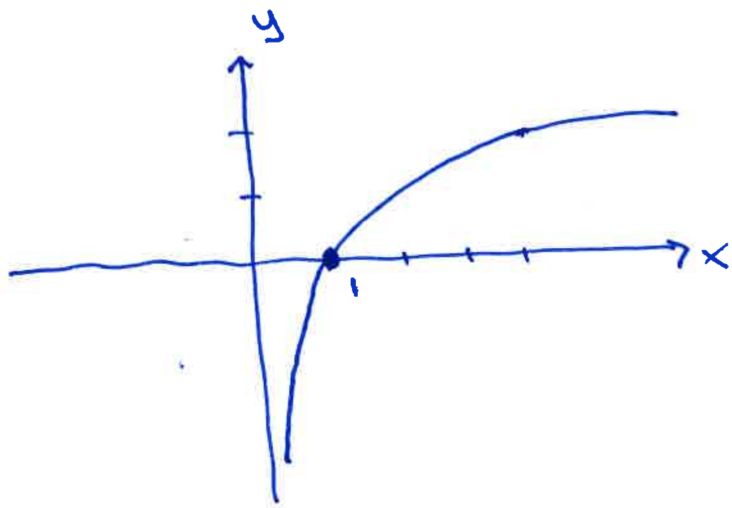


* increasing function
* $f(x) \to \infty$ as $x \to \infty$
* $f(x) \to 0$ as $x \to -\infty$

$D_f = \mathbb{R} = (-\infty, \infty)$ $\quad V_f = (0, \infty)$

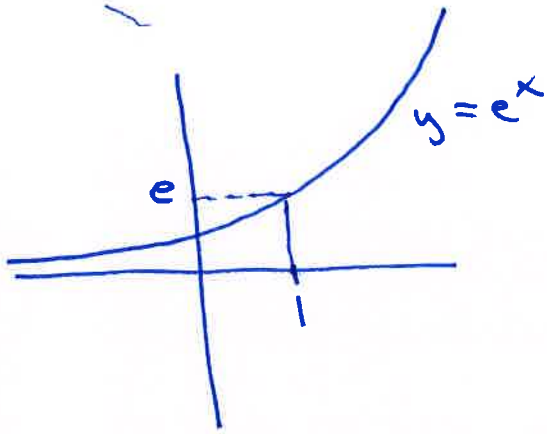
Inverse function of $f(x) = 2^x$: $\quad g(x) = \log_2(x)$

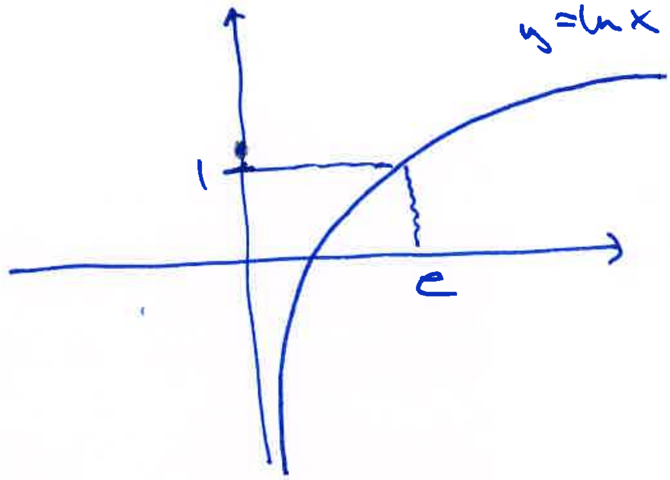
$\log_2(4) = 2$ since $f(2) = 4$



* $D_g = (0, \infty)$ $\quad V_g = \mathbb{R}$
* $g(x) \to \infty$ as $x \to \infty$
* $g(x) \to -\infty$ as $x \to 0^+$

$\underline{a=e} = 2.71828\ldots$





$\underline{\ln(x)}$: natural logarithm $= \log_e(x)$

$\underline{f(x)=a^x} \quad (a>0)$

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = \frac{a^x\cdot a^h - a^x}{h} = \frac{a^x\cdot(a^h-1)}{h}$$

$$= a^x\left(\frac{a^h-1}{a}\right) \to a^x\cdot\ln(a) = (a^x)'$$

$a=e$: $a^x\cdot\ln(a) = a^x\cdot 1 = e^x$

$$\boxed{(e^x)'=e^x}$$

Rules for powers of a:

i) $a^m\cdot a^n = a^{m+n}$

ii) $a^m : a^n = a^{m-n}$

iii) $(a^m)^n = a^{m\cdot n}$

$$e^{\ln x} = x$$

$u=\ln x \to (e^u)' = (e^{\ln x})' = (x)'$

$e^u\cdot(\ln x)' = 1$

$e^{\ln x}\cdot(\ln x)'=1$

$x\cdot(\ln x)'=1$

$(\ln x)' = 1/x$

Ex: $f(x) = x \cdot e^x = u \cdot v$

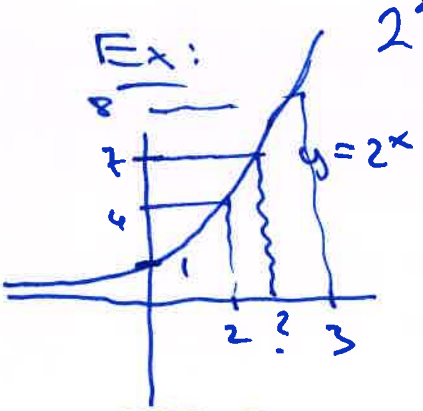
$f'(x) = u'v + uv' = 1 \cdot e^x + x \cdot e^x = \underline{\underline{(x+1)e^x}}$

$f(x) = \ln(x^2+1) = \ln(u)$, $u = x^2+1$

$f'(x) = \frac{1}{u} \cdot 2x = \underline{\underline{\frac{2x}{x^2+1}}}$

$\boxed{\begin{array}{l} (e^x)' = e^x \\ (\ln x)' = 1/x \end{array}}$

Ex:



$2^x = 7 \Rightarrow \log_2(2^x) = \log_2(7)$

$x = \underline{\underline{\log_2(7)}}$

$2^x = 7 \Rightarrow \ln(2^x) = \ln(7)$

$\frac{x \cdot \ln(2)}{\ln(2)} = \frac{\ln(7)}{\ln(2)}$

$x = \underline{\underline{\frac{\ln(7)}{\ln(2)}}}$

Rules for logarithms:

i) $\ln(x \cdot y) = \ln(x) + \ln(y)$

ii) $\ln(x/y) = \ln(x) - \ln(y)$

iii) $\ln(x^n) = n \cdot \ln(x)$

iv) $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

③ Higher derivatives and convexity

Ex:

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = (3x^2-3)' = 3 \cdot 2x - 0 = \underline{6x}$

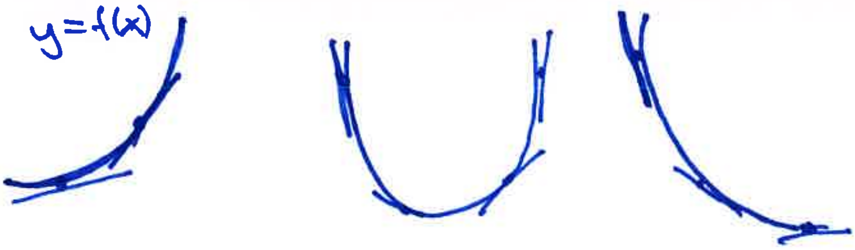
$\left(\begin{array}{l} f'''(x) = 6 \\ f^{(4)}(x) = 0 \end{array} \right)$

$\underline{f''(x) > 0}$:

tangent line of $y = f'(x)$ has positive slope

$\Updownarrow$

$f'(x)$ is increasing



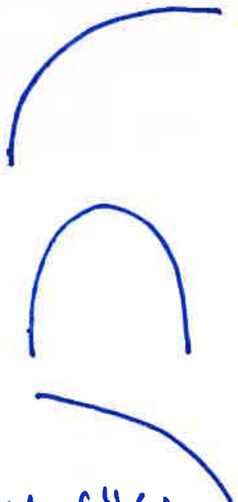
Defn:

if $f''(x) \geq 0$ for all $x \in [a,b] = I$

f is <u>convex</u> on I

$\underline{f''(x) < 0}$:

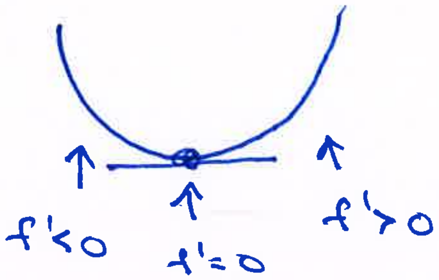
$f'(x)$ is decreasing



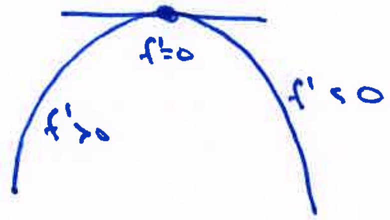
if $f''(x) \leq 0$ for $x \in I$

f is <u>concave</u> on I

If f is convex and $f'(a)=0$, then $x=a$ is (global) min



If f is concave and $f'(a)=0$, then $x=a$ is (global) max



Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = \underline{3(x-1)(x+1)}$

$f''(x) = \underline{6x}$