Plan:

1. Functions in two variables
2. Partial derivatives and the Hessian matrix
3. Unconstrained optimization
4. Lagrange problems

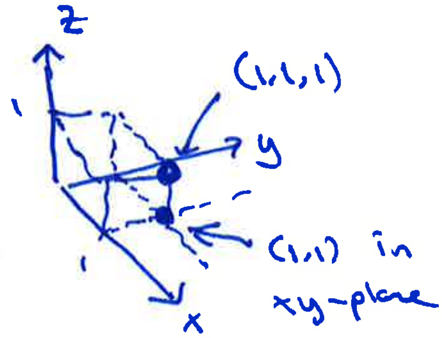
1 Functions in two variables

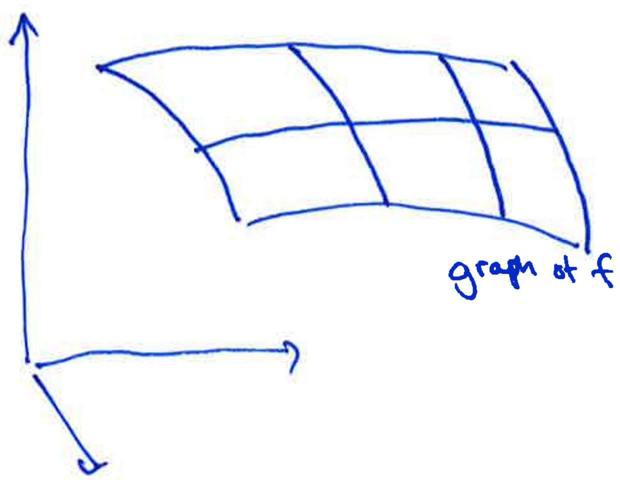
$f(x,y) =$ expression in x and y

Ex: $f(x,y) = x^3 - xy + y^2$

Graph of f: $z = f(x,y)$

Ex: $(x,y) = (1,1)$: $f(1,1) = 1^3 - 1\cdot 1 + 1^2$
$= 1$





$f(x,y) = ax + by + c$

$z = ax + by + c$ is a plane

2 Partial derivatives:

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$

$f'_y(x,y) = \lim_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$

$f'_x = (x^3 - xy + y^2)'_x$

$= 3x^2 - y + 0 = \underline{\underline{3x^2 - y}}$

$f'_y = (x^3 - xy + y^2)'_y = 0 - x \cdot 1 + 2y$

$= \underline{\underline{-x + 2y}}$

$(x,y) = (1,1)$: $f'_x(1,1) = 3 \cdot 1^2 - 1 = \underline{2}$

$f'_y(1,1) = -1 + 2 \cdot 1 = \underline{1}$

Ex: $f(x,y) = e^{-x^2 - y^2} = e^u$, $u = -x^2 - y^2$

$f'_x = e^u \cdot u'_x = e^u \cdot (-2x) = \underline{-2x\, e^{-x^2-y^2}}$

$f'_y = e^u \cdot u'_y = e^u \cdot (-2y) = \underline{-2y\, e^{-x^2-y^2}}$

Hessian matrix:

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = \underline{3x^2 - y}$ $\qquad f''_{xx} = 6x \qquad f''_{xy} = -1$

$f'_y = \underline{-x + 2y}$ $\qquad f''_{yx} = -1 \qquad f''_{yy} = 2$

Result: For all "nice" functions, $f''_{xy} = f''_{yx}$.

Hessian matrix: $H(f)(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix}$

The Hessian matrix is symmetric.

③ Unconstrained optimization

max / min $f(x,y)$

Defn: A stationary pt. for f is a point such that $f'_x = f'_y = 0$.

For "nice" functions, we have:

$$\boxed{(x^*, y^*) \text{ is max/min} \Longrightarrow \text{Stationary pt.}}$$

Ex: $f(x,y) = x^3 - xy + y^2$

$f'_x = 3x^2 - y$

$f'_y = -x + 2y$

Candidate pts = Stationary pts.

Stationary pts:

$f'_x = 3x^2 - y = 0$

$f'_y = -x + 2y = 0$

$y = 3x^2$

$-x + 2y = 0$

$-x + 2\cdot(3x^2) = 0$

$6x^2 - x = 0$

$x(6x-1) = 0$

$x = 0$ or $x = 1/6$

$y = 0$ $\quad y = 3\cdot\frac{1}{36} = \frac{1}{12}$

Stationary pts:

$(x,y) = (0,0), (1/6, 1/12)$

Defn: Let (x^*, y^*) be a stationary pt for f. It is called a

local min if $f(x^*, y^*) \leq f(x,y)$ for all (x,y) close to (x^*, y^*)

local max if $f(x^*, y^*) \geq f(x,y)$ for all (x,y) close to (x^*, y^*)

saddle pt. in all other cases

Local classification: Second derivative test

(x^*, y^*) stat. pt for f $\longrightarrow$ $A = f''_{xx}(x^*, y^*)$

$B = f''_{xy}(x^*, y^*)$

$C = f''_{yy}(x^*, y^*)$

$AC - B^2 > 0$: $A > 0$: (x^*, y^*) local min

$A < 0$: (x^*, y^*) local max

$AC - B^2 < 0$: (x^*, y^*) saddle pt

$\boxed{AC - B^2 = 0}$ we don't know

Ex: $f = x^3 - xy + y^2$ $\quad f''_{xx} = 6x \quad f''_{xy} = -1$

$f''_{yx} = -1 \quad f''_{yy} = 2$

(0,0): $A = 0 \quad B = -1 \quad C = 2$

$AC - B^2 = 0 \cdot 2 - (-1)^2 = -1$ saddle pt

(1/6, 1/12): $A = 1 \quad B = -1 \quad C = 2$

$AC - B^2 = 1 \cdot 2 - (-1)^2 = 1 > 0$ local min

Conclusion: f has no global max

(1/6, 1/12) is local min for f and could be a global min.

$f(x,y) = x^3 - xy + y^2$ $\quad \begin{cases} y = 0 \\ x \text{ large negative number} \end{cases}$ no global min

Second derivative test:

(x^*, y^*) Stat. pt. for f $\longrightarrow$ $H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$\det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \underline{AC - B^2}$

$\det H(f)(x^*, y^*) > 0$ — $A > 0$: local min; $A < 0$: local max

$\det H(f)(x^*, y^*) < 0$: saddle pt.

$\det H(f)(x^*, y^*) = 0$: no conclusion

$\underline{A = f''_{xx}(x^*, y^*)}$:

$A > 0$

$A < 0$

$\underline{C = f''_{yy}(x^*, y^*)}$:

$C > 0$

$C < 0$

Ex: $A > 0$, $C < 0$

Ex: $f(x,y) = x^3 - 3xy + y^3$

1) Candidate pts = stationary pts
2) Local classific. using second derivative test
3) Conclude

$f'_x = 3x^2 - 3y = 0$
$f'_y = -3x + 3y^2 = 0$
FOC = first order conditions

$\frac{3x^2}{3} = \frac{3y}{3} \Rightarrow \underline{y = x^2}$

$-3x + 3(x^2)^2 = 0$
$-3x + 3x^4 = 0$
$-3x(1 - x^3) = 0$

$-3x = 0$ or $1 - x^3 = 0$

$\underline{x = 0}$, $\underline{y = 0}$

$x^3 = 1$
$x = \sqrt[3]{1} = 1$
$\underline{x = 1}$, $\underline{y = 1}$

Stat. pts: $(0,0)$, $(1,1)$
$f = 0$, $f = -1$

$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

$\det = 0 - 9 = -9 < 0$

$(0,0)$ saddle pt

$(1,1)$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

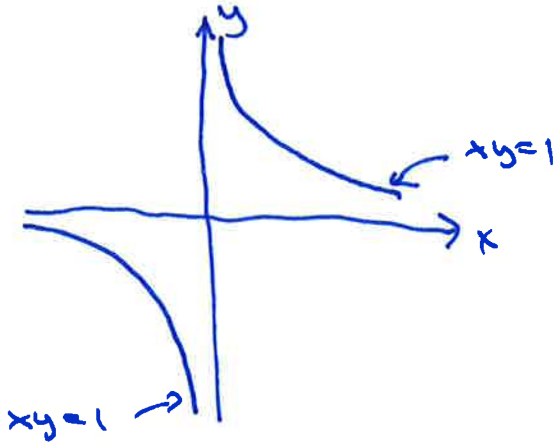
$\det = 36 - 9 > 0$ $A > 0$

$(1,1)$ local [redacted] min

4 Lagrange problems

max/min $f(x,y)$ when $\underbrace{g(x,y) = a}_{\text{equality constraint}}$

Ex: min $\underbrace{f(x,y) = x^2+y^2}_{\text{objective fn.}}$ when $\underbrace{xy=1}_{\text{equality constraint}}$



Lagrange's method:

$$\mathcal{L} = f(x,y) - \lambda g(x,y)$$
$$= x^2+y^2 - \lambda \cdot xy$$

$xy=1 \Rightarrow y = 1/x$

$$\mathcal{L} = x^2+y^2 - \lambda(xy-1)$$

$xy = 1$

$xy - 1 = 0$

Candidate pts:

FOC: (first order conditions)
$$\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$$

C: $g = a$

$$\begin{aligned} \mathcal{L}'_x &= \boxed{2x - \lambda y = 0} \\ \mathcal{L}'_y &= \boxed{2y - \lambda x = 0} \end{aligned} \Bigg\} \text{FOC}$$
$$\boxed{xy = 1} \Big\} \text{C}$$

$\boxed{\text{Solutions of FOC+C = candidate pts}}$

$$\boxed{\begin{aligned} 2x - \lambda y &= 0 \\ 2y - \lambda x &= 0 \\ xy &= 1 \end{aligned}}$$

(1) $\frac{2x}{2} = \frac{\lambda y}{2}$ $\quad \underline{x = \frac{\lambda y}{2}}$

(2) $2y - \lambda x = 0$

$2y - \lambda \cdot \frac{\lambda y}{2} = 0 \quad | \cdot 2$

$4y - \lambda^2 y = 0$

$y(4-\lambda^2) = 0$

$\underline{y=0}$ or $\underline{4-\lambda^2=0}$

$y=0$: $x \cdot 0 = 1$ not possible $\Downarrow$ no solutions

$\lambda^2 = 4$

$\lambda = \pm\sqrt{4} = \underline{\pm 2}$

$\underline{\lambda = 2}$

$x = \frac{2y}{2} = y$

(3) $xy = 1$

$y^2 = 1$

$y = \pm 1$

$(x, y; \lambda) =$

$(1, 1; 2)$,

$(-1, -1; 2)$

$\underline{\lambda = -2}$

$x = \frac{-2y}{2} = -y$

(3) $xy = 1$

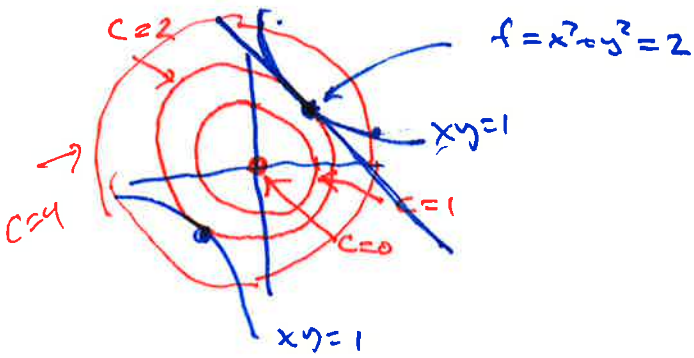
$-y \cdot y = 1$

$y^2 = -1$

$\underline{\text{no solutions}}$

Candidate pts: $(x, y; \lambda) = \underline{(1, 1; 2)}$, $\underline{(-1, -1; 2)}$

$f(1,1) = 2$ $\qquad f(-1,-1) = 2$



Level curves for f: $f(x,y) = c$

$x^2 + y^2 = c$

$c = 1$: $x^2 + y^2 = 1$

$c = 0$: $x^2 + y^2 = 0$

$c = 2$: $x^2 + y^2 = 2$

$c = 4$: $x^2 + y^2 = 4$

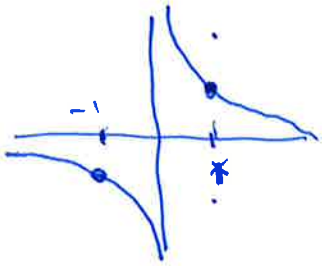
Alternative: min $f = x^2+y^2$ when $xy=1$

$$\Updownarrow$$

$$y = 1/x$$

$$f(x,y) = x^2 + (1/x)^2$$

$$f(x) = x^2 + 1/x^2, \quad x \neq 0$$



$$f'(x) = 2x - 2x^{-3} = 2x - 2\frac{1}{x^3}$$

$$= \frac{2x \cdot x^3 - 2}{x^3} = \frac{2x^4-2}{x^3} = 0$$

$$f'(x) = \frac{2(x^4-1)}{x^3}$$

$$= \frac{2(x^2-1)(x^2+1)}{x^3}$$

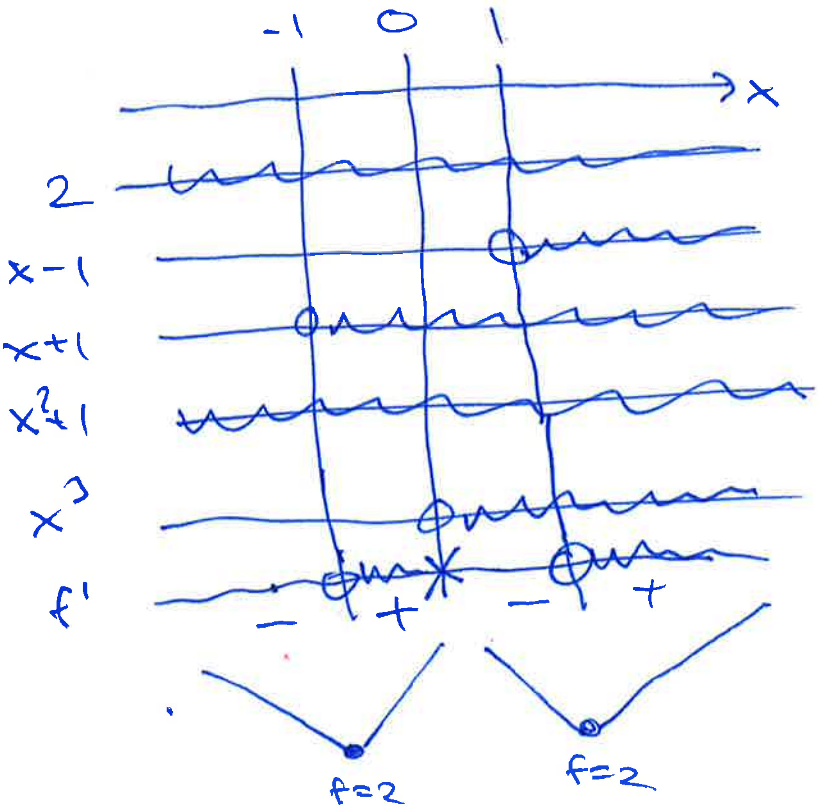
$$= \frac{2(x-1)(x+1)(x^2+1)}{x^3}$$

$$2x^4 - 2 = 0$$

$$x^4 = 1$$

$$x = \pm\sqrt[4]{1} = \pm 1$$

$$\underline{x = \pm 1}$$



Global min: $f=2$

in $x=1$ and $x=-1$

$$\Updownarrow \qquad \Updownarrow$$

$$(1,1) \qquad (-1,-1)$$

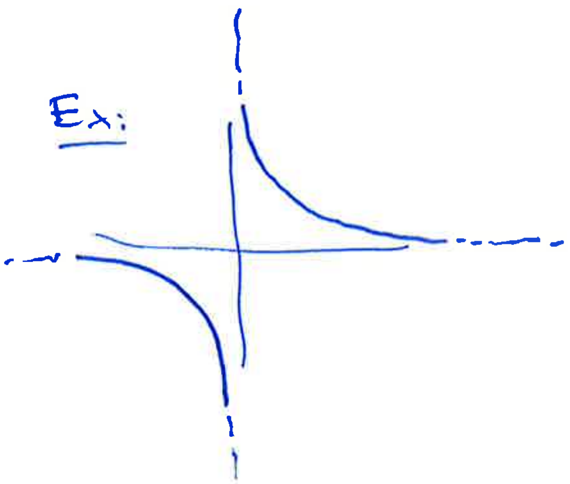
Extreme value theorem:

A continuous function on a closed and bounded region has a max and a min.

max/min $f(x,y)$ when $g(x,y) = a$

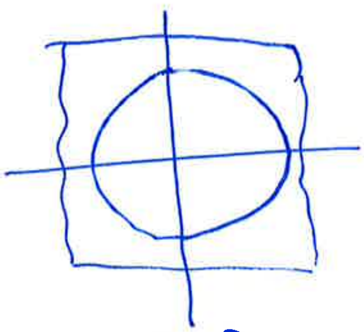
1) f cont: ok
2) $D = \{(x,y) : g(x,y) = a\}$ closed: ok
3) $D =$ — " — bounded: must check!

Ex:



not bounded

Ex:



$x^2 + y^2 = 4$

bounded

The first order leading principal minor is $F_{xx} = 6x$ and the second order leading principal minor is $\det D^2F(\mathbf{x}) = -36xy - 81$. At $(0,0)$, these two minors are 0 and -81, respectively. Since the second order leading principal minor is negative, $(0,0)$ is a saddle of F — neither a max nor a min. At $(3,-3)$, these two minors are 18 and 243. Since these two numbers are positive, $D^2F(3,-3)$ is positive definite and $(3,-3)$ is a strict local min of F.

Notice that $(3,-3)$ is not a *global* min, because at the point $(0,n)$, $F(0,n) = -n^3$, which goes to $-\infty$ as $n \to \infty$.

EXERCISES stationary pts.

17.1 For each of the following functions defined on $\mathbf{R}^2$, find the critical points and classify these as local max, local min, saddle point, or "can't tell":

$a)$ $x^4 + x^2 - 6xy + 3y^2$, $b)$ $x^2 - 6xy + 2y^2 + 10x + 2y - 5$,
$c)$ $xy^2 + x^3y - xy$, $d)$ $3x^4 + 3x^2y - y^3$.

17.2 For each of the following functions defined on $\mathbf{R}^3$, find the critical points and classify them as local max, local min, saddle point, or "can't tell":
$a)$ $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$,
$b)$ $(x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$.

17.4 GLOBAL MAXIMA AND MINIMA

The first and second order sufficient conditions of the last section will find all the local maxima and minima of a differentiable function whose domain is an open set in $\mathbf{R^n}$. As Example 17.2 illustrates, these conditions say nothing about whether or not any of these local extrema is a *global* max or min. In this section, we will discuss sufficient conditions for global maxima and minima of a real-valued function on $\mathbf{R^n}$.

The study of one-dimensional optimization problems in Section 3.5 put forth two conditions for a critical point x^* of f to be a global max (or min), when f is a C^2 function defined on a connected interval I of $\mathbf{R}^1$:

(1) x^* is a local max (or min) and it's the only critical point of f in I; or
(2) $f'' \leq 0$ on all of I (or $f'' \geq 0$ on I for a min), that is, f is a concave function on I (or f is a convex function for a min).

Condition 1 does not work in higher dimensions, as the function F whose level sets are pictured in Figure 17.1 illustrates. The point A in Figure 17.1 is a local max of F in the open set U. Even though A is the only critical point of F in U, the function F takes on a higher value at point B.

Problems for Lecture 6

1. Find all stationary points and classify them

a) $f(x,y) = e^{xy}$

b) $f(x,y) = e^{x-2y}$

c) $f(x,y) = \sqrt{x^2+y^2+1}$

d) $f(x,y) = x\ln x + y\ln y$

e) ✱ $f(x,y) = x\ln(y) - y\ln(x)$ (✱ Difficult.)

2. Solve the Lagrange problems

a) $\max/\min\ f(x,y) = 3x+4y$ when $x^2+y^2=25$

b) $\max\ f(x,y) = y$ when $x^2+y^3=0$

c) $\min\ f(x,y) = 3x^2+4y^2$ when $xy=1$

Solutions for Lecture 6

1. a) $f'_x = ye^{xy}$ $\quad f''_{xx} = y^2 e^{xy}$ $\quad f''_{xy} = (1+xy)e^{xy}$

$f'_y = xe^{xy}$ $\quad f''_{yy} = x^2 e^{xy}$

$f'_x = f'_y = 0$

$y = x = 0 \Rightarrow$ Stat: $(0,0)$

$H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$AC - B^2 = -1 < 0$ Saddle pt

b) $u = x - 2y$

$f'_x = e^u \cdot 1$ $\quad f''_{xx} = e^u \cdot 1$ $\quad f''_{xy} = e^u \cdot 1 \cdot (-2)$

$f'_y = e^u \cdot (-2)$ $\quad f''_{yy} = e^u (-2)^2$

$f'_x = f'_y = 0$

$e^{x-2y} = 0 \Rightarrow$ no stat. pts

impossible

~~c) $u = x^2 + y^2 + 1$~~

~~$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{u}}$~~

~~$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{y}{\sqrt{u}}$~~

~~$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $f'_x = f'_y = 0$, $x = y = 0 \Rightarrow$ ($u = 1 \neq 0$) Stat. pts: $(0,0)$~~

~~$f''_{xx} = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{2x}{2\sqrt{u}}\right) \cdot 2\sqrt{u}}{u \cdot 2\sqrt{u}} = \frac{2u - x^2}{2u\sqrt{u}} = \frac{x^2 + y^2 + 1 - x^2}{u\sqrt{u}} = \frac{y^2+1}{u\sqrt{u}}$~~

~~$f''_{xy} = \frac{-x \cdot \frac{1}{2\sqrt{u}} \cdot 2y}{u} = \frac{-x^2}{u\sqrt{u}}$~~

~~$f''_{yy} = \frac{x^2+1}{u\sqrt{u}}$ $\leftarrow$ symmetry $f(y,x) = f(x,y)$, $f''_{yy}(x,y) = f''_{xx}(y,x)$~~

~~$H(f)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (0,0)$ is local min~~

~~$AC - B^2 = 1 > 0$, $A = 1 > 0$~~

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c) $f(x,y) = \sqrt{u}$ with $u = x^2+y^2+1$

$$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{u}} = \underline{\frac{x}{\sqrt{u}}}$$

$$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{2y}{2\sqrt{u}} = \underline{\frac{y}{\sqrt{u}}}$$

$f'_x = f'_y = 0$: $\frac{x}{\sqrt{u}} = 0 \Rightarrow x = 0$

$\frac{y}{\sqrt{u}} = 0 \Rightarrow y = 0$

$(u = \sqrt{1} \neq 0)$

$\Rightarrow$ Stat. pts: $(x,y) = \underline{(0,0)}$

$$f''_{xx} = \left(\frac{x}{\sqrt{u}}\right)'_x = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}x\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}}$$

$$= \frac{u - x^2}{u\sqrt{u}} = \frac{\cancel{x^2} + y^2 + 1 - \cancel{x^2}}{u\sqrt{u}} = \underline{\frac{y^2+1}{u\sqrt{u}}}$$

$$f''_{xy} = \left(\frac{x}{\sqrt{u}}\right)'_y = \frac{\left(0 \cdot \sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \underline{\frac{-xy}{u\sqrt{u}}}$$

$$f''_{yy} = \left(\frac{y}{\sqrt{u}}\right)'_y = \frac{\left(1 \cdot \sqrt{u} - y \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \frac{u - y^2}{u\sqrt{u}} = \underline{\frac{x^2+1}{u\sqrt{u}}}$$

$$H(f)(0,0) = \begin{pmatrix} 1/1\sqrt{1} & 0/1\sqrt{1} \\ 0/1\sqrt{1} & 1/1\cdot\sqrt{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\left.\begin{array}{l} \det H(f)(0,0) = AC - B^2 = 1 > 0 \\ A = 1 > 0 \end{array}\right\} \Rightarrow (0,0)$ is a $\underline{\text{local min}}$

d) $f'_x = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$

$f'_y = \ln y + 1$

Stat. pts:

$\ln x + 1 = \ln y + 1 = 0$

$x = y = e^{-1} \Rightarrow (x,y) = (e^{-1}, e^{-1})$

$f''_{xx} = 1/x \quad f''_{xy} = 0 \quad f''_{yy} = 1/y$

$H(f)(e^{-1},e^{-1}) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \Rightarrow (e^{-1},e^{-1})$ is local min

$AC - B^2 = e^2 > 0, \; A = e > 0$

(*) = Difficult

e) (*) $f'_x = \ln y - y\cdot\frac{1}{x} = \ln y - \frac{y}{x} = 0$

$f'_y = \cancel{\ln x}\, x\cdot\frac{1}{y} - \ln x = \frac{x}{y} - \ln x = 0$

Stat. pts:

$\ln y = \frac{y}{x} \Rightarrow x = \frac{y}{\ln y} \Rightarrow \ln\left(\frac{y}{\ln y}\right) = \frac{y/\ln y}{y} = \frac{1}{\ln y}$

$\ln x = \frac{x}{y}$

$\ln y \cdot \ln\left(\frac{y}{\ln y}\right) = 1$

$\boxed{\ln y \cdot (\ln y - \ln(\ln y)) = 1}$

$u(y) = \ln(y)\,(\ln y - \ln(\ln y))$

$u' = \frac{1}{y}(\ln y - \ln(\ln y)) + \ln y \cdot \left(\frac{1}{y} - \frac{1}{\ln y}\cdot\frac{1}{y}\right)$

$= \frac{\ln y - \ln(\ln y) + \ln y - 1}{y}$

$= \frac{2\ln y - \ln(\ln y) - 1}{y}$

To check if $u = 1$ has solutions, find out when u is inc. / dec.

⇓

look at sign of u'

$u' = 0$:

$2\ln y - \ln(\ln y) = 1$

$\ln\left(\frac{y^2}{\ln y}\right) = 1$

$\frac{y^2}{\ln y} = e$

$v = \frac{y^2}{\ln y}$

$v' = \frac{2y\ln y - y^2\frac{1}{y}}{(\ln y)^2}$

$= \frac{y(2\ln y - 1)}{(\ln y)^2}$

To check if $v = e$ has solutions, find out where v is inc. / dec.

$\Rightarrow$ look at sign of v'.

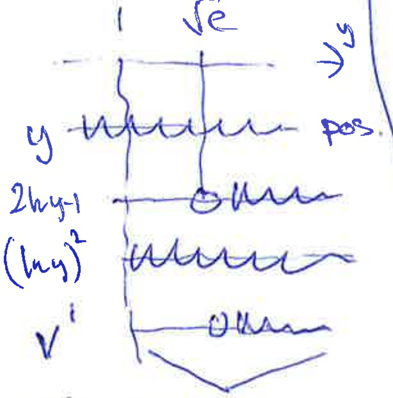
$v' = 0$:

$y\cdot(2\ln y - 1) = 0$

$2\ln y - 1 = 0$

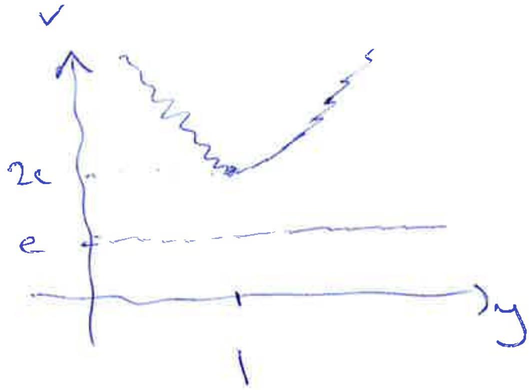
$\ln y = \frac{1}{2}$

$y = e^{1/2} = \sqrt{e}$



min for v:

$y = \sqrt{e} \Rightarrow v = \frac{e}{1/2} = 2e > e$



$v(y) = \frac{y^2}{\ln y}$

$v = e$ no solution

$u' = 0$ no solution

$u' = \frac{2\ln y - \ln(\ln y) - 1}{y}$

$y > 1$: $y > 0$, $2\ln y - \ln(\ln y) - 1$ const. sign since it is never zer

$y = e \rightsquigarrow 2 - \ln 1 - 1 = 1 > 0$

$\Downarrow$

$u' > 0$ for all $y > 1$

u inc. function on $1 < y < \infty$

$y = e$ is a solution since $\ln e\,(\ln(e) - \ln(\ln e)) = 1 \cdot (1 - 0) = 1$

u increasing fn.

$\Downarrow$

$u = 1$ has at most one solution

$\Downarrow$

$y = e$ only solution of $u = 1$

$x = \frac{y}{\ln y} = \frac{e}{\ln e} = e$

$\Downarrow$

$(x, y) = (e, e)$ unique stat. pt. of f.

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$

$$H(f)(e,e) = \begin{pmatrix} 1/e & 0 \\ 0 & 1/e \end{pmatrix}$$

$AC - B^2 = 1/e^2 > 0$

$A = 1/e > 0$

$\Downarrow$

$(x, y) = (e, e)$ is local min

2.

a) $\mathcal{L} = 3x + 4y - \lambda \cdot (x^2 + y^2)$

$$\text{FOC} \begin{cases} \mathcal{L}'_x = 3 - \lambda \cdot 2x = 0 & \Rightarrow \quad x = \dfrac{3}{2\lambda} \\ \mathcal{L}'_y = 4 - \lambda \cdot 2y = 0 & \Rightarrow \quad y = \dfrac{4}{2\lambda} \end{cases}$$

$$C \; \{ \quad x^2 + y^2 = 25$$

$$x^2 + y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 25$$

$$\frac{9 + 16}{4\lambda^2} = 25$$

$$\frac{25}{4\lambda^2} = 25$$

$$4\lambda^2 = 1$$

$$\lambda^2 = 1/4$$

$$\underline{\lambda = \pm 1/2}$$

$\underline{\lambda = 1/2}$: $x = 3,\ y = 4$

$\Downarrow$

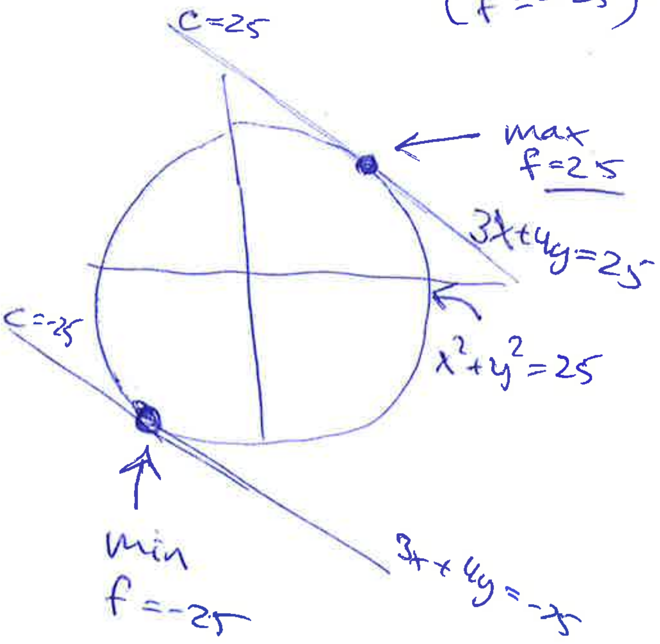
$\underline{(x, y; \lambda) = (3, 4; 1/2)}$

$(f = 25)$

$\underline{\lambda = -1/2}$: $x = -3,\ y = -4$

$\underline{(x, y; \lambda) = (-3, -4; -1/2)}$

$(f = -25)$



inc. values of c

means

lines = level curves move

up and to the right

b) max y when $x^2+y^3=0$

$h = y - \lambda\cdot(x^2+y^3)$

FOC $\left\{ \begin{array}{l} h'_x = -\lambda\cdot 2x = 0 \\ h'_y = 1-\lambda\cdot 3y^2 = 0 \end{array} \right.$ $\Rightarrow$ $\underline{\lambda = 0}$ or $\underline{x=0}$

C $\{ \quad x^2+y^3=0$

$\lambda=0$: $1=0$ $\underline{\text{imp.}}$

$x=0 \Downarrow$ $x^2+y^3=0 \Rightarrow \underline{y=0}$

$1-\lambda\cdot 3y^2=0 \Rightarrow 1=0$ $\underline{\text{imp.}}$

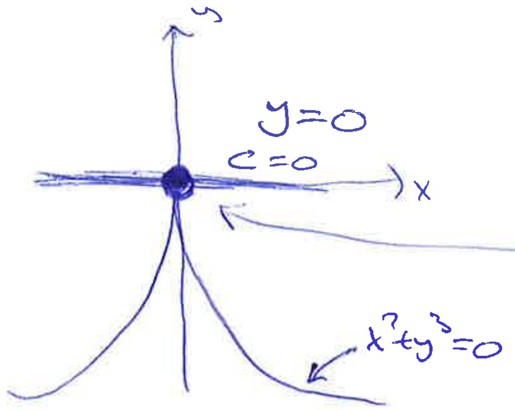
no solution of $\underline{\text{FOC+C}}$.

$$\boxed{\begin{array}{l} g'_x = 2x = 0 \\ g'_y = 3y^2 = 0 \\ x^2+y^3=0 \end{array}} \quad \left. \begin{array}{l} x=0 \\ y=0 \\ x=y=0 \ \underline{\text{ok}}. \end{array} \right\} \ (0,0) \text{ is adm. pt.}$$

with $g'_x = g'_y = 0$

$\Downarrow$

$\underline{\text{can be max}}$



inc. values of c means level curve $y=c$ moves $\underline{\text{up}}$.

$\underline{\underline{\max = 0}}$

c) $\min f = 3x^2+4y^2$ when $xy=1$

$$\mathcal{L} = 3x^2+4y^2 - \lambda \cdot xy$$

FOC $\left\{ \begin{array}{l} \mathcal{L}'_x = 6x - \lambda y = 0 \\ \mathcal{L}'_y = 8y - \lambda x = 0 \end{array} \right.$

C $\{ \quad xy = 1$

① $x = \dfrac{\lambda y}{6}$

② $8y - \lambda \cdot \left(\dfrac{\lambda y}{6}\right) = 0 \quad | \cdot 6$

$48y - \lambda^2 y = 0$

$y(48-\lambda^2) = 0$

$\underline{y=0}$ or $\lambda^2 = 48$

$\underline{\lambda = \pm\sqrt{48}}$

$y=0$	$\lambda = \sqrt{48}$	$\lambda = -\sqrt{48}$
③ $xy=1$, $x \cdot 0 = 1$ imp. no soln.	① $x = \dfrac{\sqrt{48}}{6}y$	① $x = -\dfrac{\sqrt{48}}{6}y$
	③ $xy = \dfrac{\sqrt{48}}{6}y \cdot y = 1$	③ $xy = -\dfrac{\sqrt{48}}{6}y \cdot y = 1$
	$y^2 = \dfrac{6}{\sqrt{48}} = \dfrac{\cancel{2} \cdot 3}{\cancel{\sqrt{4}} \cdot \sqrt{12}} = \dfrac{3}{\sqrt{12}} = \dfrac{\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{4}} = \sqrt{3/4}$	$y^2 = -\dfrac{6}{\sqrt{48}}$ imp. no soln.
	$y = \pm\sqrt[4]{3/4}$	
	$x = \pm\sqrt[4]{3/4} \cdot \dfrac{\sqrt{48}}{6} = \pm\sqrt[4]{4/3}$	

$\pm\sqrt[4]{3/4} \cdot \sqrt{\dfrac{48}{36}}$

$= \pm\sqrt[4]{\dfrac{3}{4}} \cdot \sqrt[4]{\left(\dfrac{4}{3}\right)^2}$

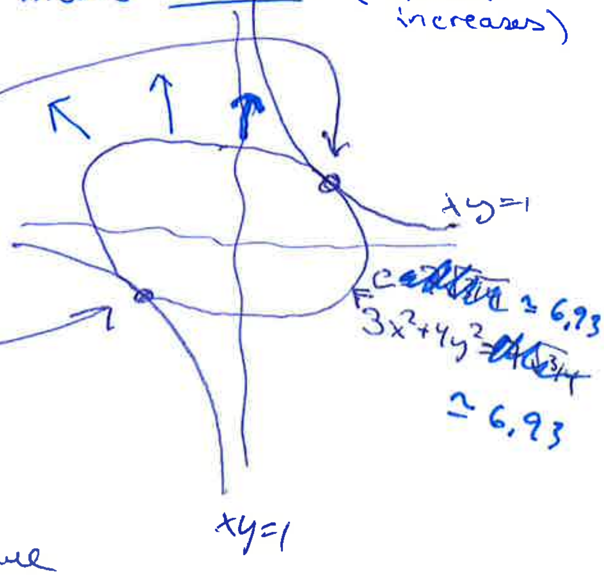
$= \pm\sqrt[4]{4/3}$

Pts:

$\left(\sqrt[4]{4/3}, \sqrt[4]{3/4}; \sqrt{48}\right)$

$\left(-\sqrt[4]{4/3}, -\sqrt[4]{3/4}; \sqrt{48}\right)$

inc. values of c means the level curve = ellipse $3x^2+4y^2=c$ moves outwords ("radius" increases)



$f = 3\cdot\sqrt{4/3} + 4\cdot\sqrt{3/4}$

$\simeq 6.93$ min pt./value