Plan:

① Application of the derivative to analyze and graph functions

② Integration

Reading:

[ME] Ch. 13-14, Appendix A4

(See also [DE], Appendix on Integr.)

[redacted]

① Applications of the derivative

Ex: $f(x) = e^{2x} + e^{-3x}$

$$g(x) = \frac{3}{5}\underline{\ln(1+x)} + \frac{2}{5}\underline{\ln(1-x)}$$

Recall:

$$\left(e^{u(x)}\right)' = e^{u(x)} \cdot u'(x)$$

$$\left(\ln(u(x))\right)' = \frac{1}{u(x)} \cdot u'(x)$$

$$f'(x) = e^{2x} \cdot 2 + e^{-3x} \cdot (-3)$$

$$= \underline{2e^{2x} - 3e^{3x}}$$

$$g'(x) = \frac{3}{5} \cdot \frac{1}{1+x} \cdot 1 + \frac{2}{5} \cdot \frac{1}{1-x} \cdot (-1)$$

$$= \frac{3/5}{1+x} - \frac{2/5}{1-x} = \underline{\frac{1}{5}\left(\frac{3}{1+x} - \frac{2}{1-x}\right)}$$

How to use the derivative to analyze f:

($f(x)$: a function in one variable)

a) **$f'(x)$:** $f'(x)>0$ in the interval (a,b) $\Rightarrow$ f is increasing in $[a,b]$

$f'(x)<0$ — " — $\Rightarrow$ f is decreasing in $[a,b]$

$f'(x)=0$: x is called a stationary pt.

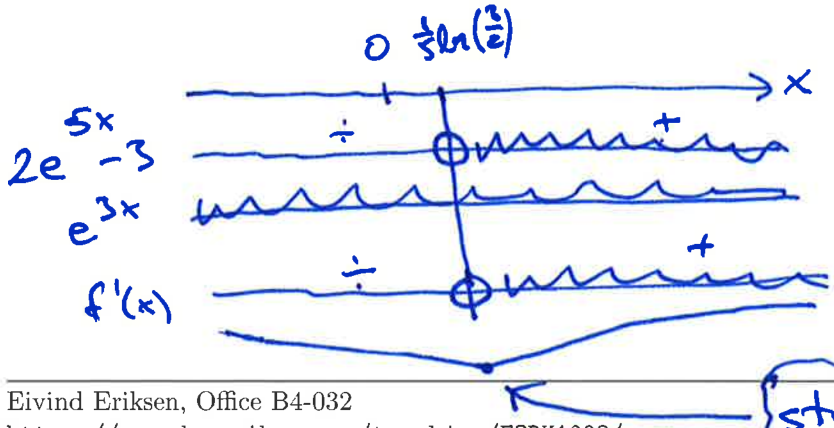
b) **$f''(x)$:** $f''(x)>0$ in the interval (a,b) $\Rightarrow$ f is convex in $[a,b]$

$f''(x)<0$ — " — $\Rightarrow$ f is concave in $[a,b]$

Ex: $f(x) = e^{2x} + e^{-3x}$

$f'(x) = 2e^{2x} - 3e^{-3x}$

$$= 2e^{2x} - \frac{3}{e^{3x}} = \frac{2e^{2x}\cdot e^{3x} - 3}{e^{3x}} = \frac{2e^{5x}-3}{e^{3x}}$$



$2e^{5x} - 3 = 0$

$2e^{5x} = 3$

$e^{5x} = 3/2$

$5x = \ln(3/2)$

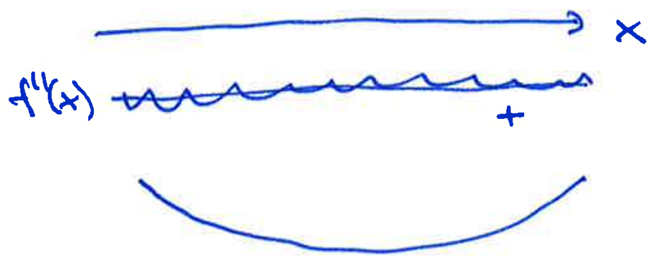
$x = \frac{1}{5}\ln(3/2)$

stationary pt.
minimum

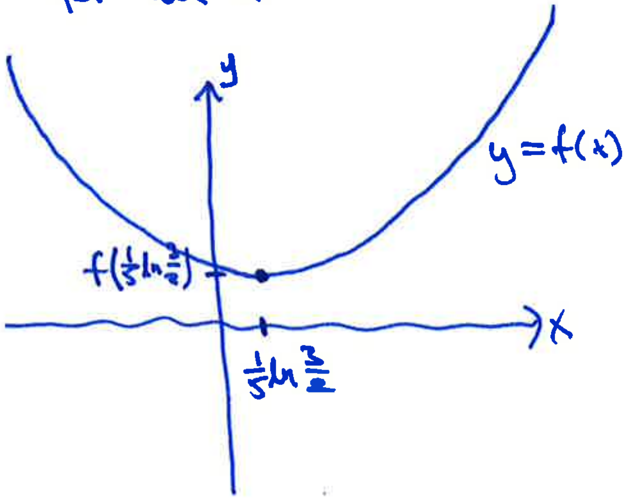
$$f'(x) = 2e^{2x} - 3e^{-3x}$$

$$f''(x) = 2(e^{2x} \cdot 2) - 3 \cdot (e^{-3x} \cdot (-3))$$

$$= 4e^{2x} + 9e^{-3x} > 0 \quad \text{for all } x$$



f is convex

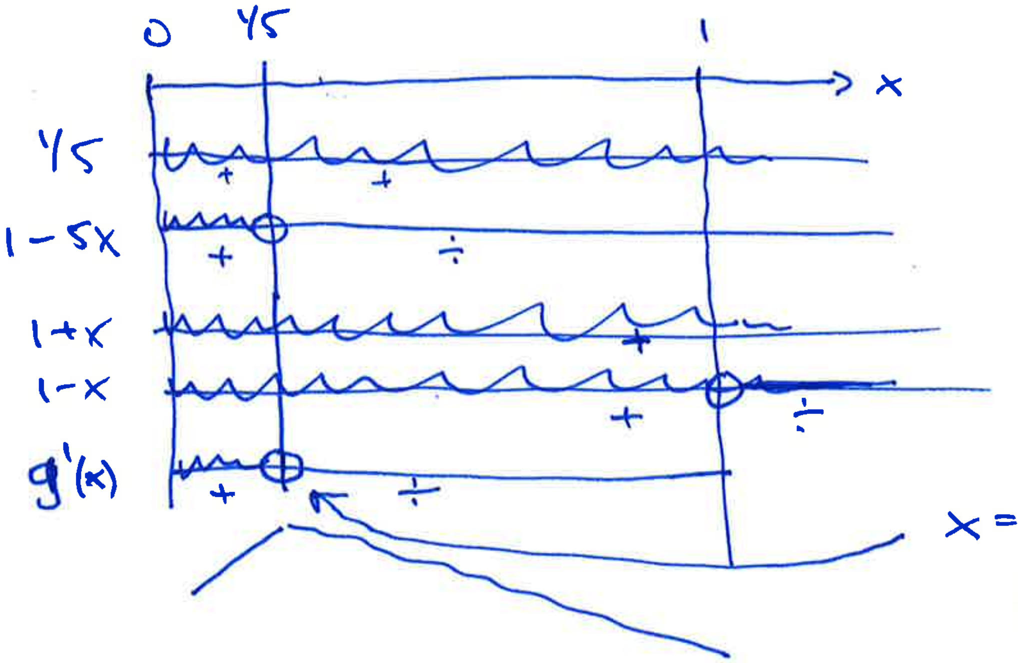


Ex: $g(x) = 0.6 \ln(1+x) + 0.4 \ln(1-x), \quad x \in [0,1)$

$$= \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$$

$$g'(x) = \frac{1}{5}\left(\frac{3}{1+x} - \frac{2}{1-x}\right) = \frac{1}{5}\left(\frac{3\,(1-x)}{(1+x)(1-x)} - \frac{2\,(1+x)}{(1-x)(1+x)}\right)$$

$$= \frac{1}{5} \cdot \frac{3-3x-2-2x}{(1+x)(1-x)} = \frac{1}{5} \cdot \frac{1-5x}{(1+x)(1-x)}$$



$x = \frac{1}{5} = 0.2$ stationary pt

global maximum for g

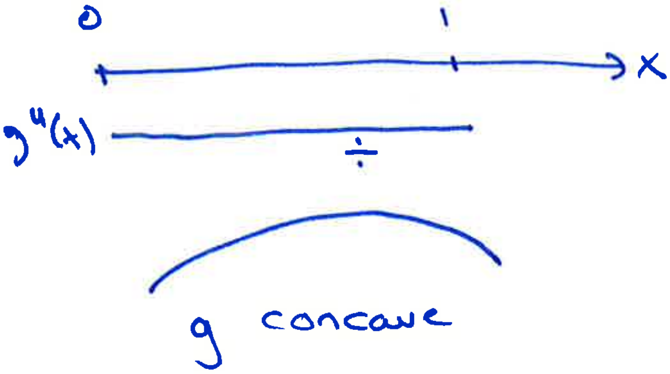
$$g'(x) = \frac{1}{5}\left(\frac{3}{1+x} - \frac{2}{1-x}\right)$$

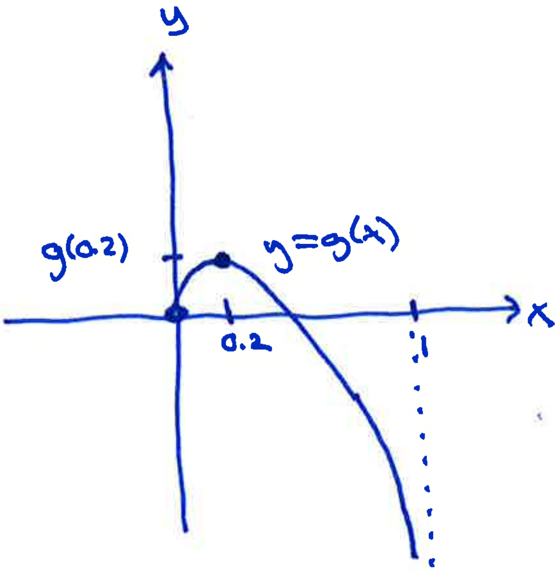
$$g''(x) = \frac{1}{5}\left(\frac{3}{1+x} - \frac{2}{1-x}\right)' = \frac{1}{5}\left(3(1+x)^{-1} - 2(1-x)^{-1}\right)'$$

$$= \frac{1}{5}\left(3\cdot(-1)(1+x)^{-2}\cdot 1 - 2\cdot(-1)\cdot(1-x)^{-2}\cdot(-1)\right)$$

$$= \frac{1}{5}\left(\frac{-3}{(1+x)^2} - \frac{2}{(1-x)^2}\right) = -\frac{1}{5}\left(\frac{3}{(1+x)^2} + \frac{2}{(1-x)^2}\right)$$

< 0 for all $x \in [0,1)$





$$g(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$$

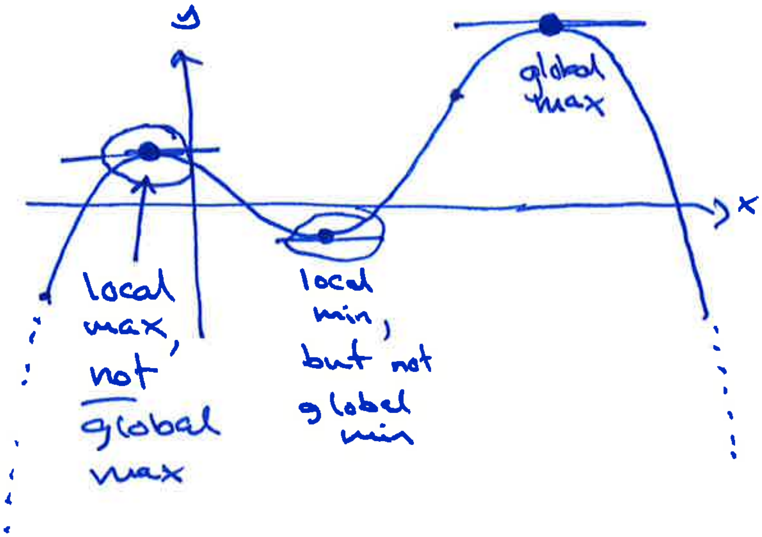
$$g(0.2) = 0.6\cdot\ln(1.2) + 0.4\ln(0.8)$$

Optimization for functions in one variable

max/min $f(x)$

Local max/min

Global max/min



no global min.

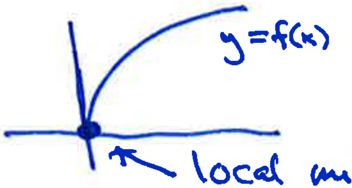
Result:

If x^* is a local max/min, then it is either

- i) stationary pt : $f'(x^*)=0$
- ii) critical pt but not stationary : $f'(x^*)$ is not defined
- iii) boundary pt.

candidate pts for max/min

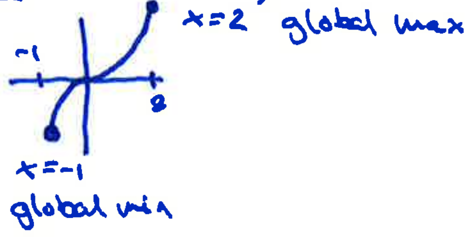
Ex: $f(x)=\sqrt{x}$, $x \geq 0$



local min & global min
critical pt, boundary pt.

$f'(x)=\frac{1}{2\sqrt{x}}$, $x>0$

Ex: $f(x)=x^3$, $x \in [-1,2]$



Second derivative test:

If x^* is a stationary pt for f, then:

$f''(x^*) > 0 \Rightarrow x^*$ is a local min

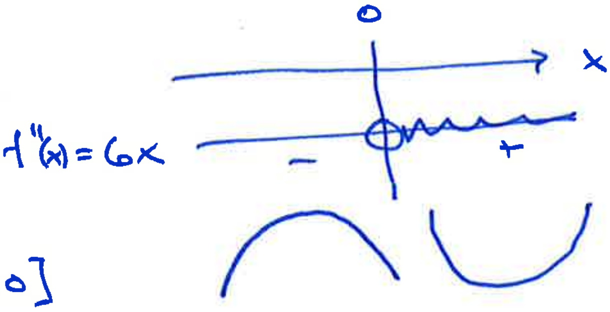
$f''(x^*) < 0 \Rightarrow x^*$ is a local max

Ex:

$f(x) = x^3 - 3x + 2$

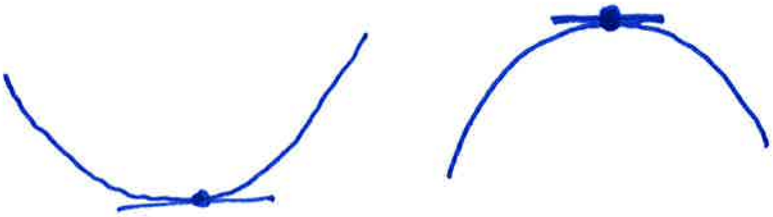
$f'(x) = 3x^2 - 3$

$f''(x) = \underline{6x}$



f is concave on $(-\infty, 0]$

f is convex " $[0, \infty)$



Convex optimization:

Result:

If f is convex, then any stationary pt. of f is a global min.

If f is concave, then any stationary pt. of f is a global max.

② Integration

$f(x)$: fn. in one variable

Defn: $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Ex: $f(x) = 2x$

$F(x) = x^2$ is an antiderivative of f

$F(x) = x^2+1$ — " —

$F(x) = x^2 + C$ is called the general antiderivative of $f(x)$.

Notation: Indefinite integral

$$\int \underbrace{f(x)}\, dx = \underbrace{F(x) + C}$$

integration sign

x is the integration variable

the general antiderivative of $f(x)$

integrand, the function that we integrate

$$\int 2x\, dx = x^2 + C$$

C = integration constant (means any constant)

$$(x^n)' = n \cdot x^{n-1}$$

Integration rules:

i) $\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + C$ for all $n \neq -1$

ii) $\int \frac{1}{x} dx = \ln|x| + C$

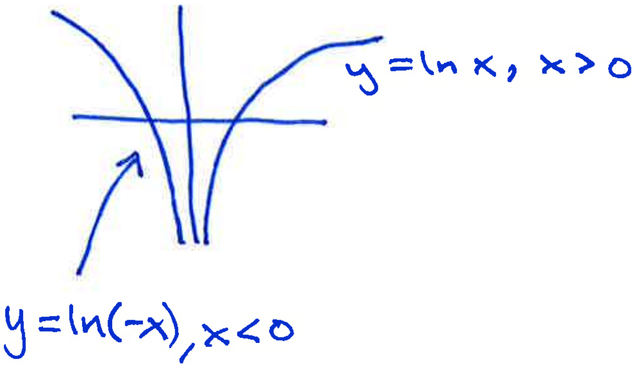
iii) $\int e^x dx = e^x + C$

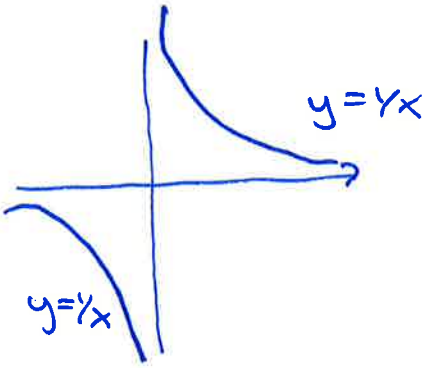
iv) $\int u(x) \pm v(x)\, dx = \int u(x)\, dx \pm \int v(x)\, dx$

v) $\int c \cdot u(x)\, dx = c \cdot \int u(x) dx$

Explanation: $\boxed{\int \frac{1}{x}\,dx} = \ln|x| + C$

$(\ln x)' = \frac{1}{x}$





$(\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

absolute value of x

$$\int \frac{1}{x}\,dx = \left\{ \begin{array}{ll} \ln(x) + C, & x > 0 \\ \ln(-x) + C, & x < 0 \end{array} \right\} = \ln|x| + C$$

Ex:

$$\int x^4 dx = \frac{1}{5}x^5 + C = \underline{\frac{x^5}{5} + C}$$

$$\int 1 - 2x + 3x^2 dx = \underline{x - x^2 + x^3 + C}$$

$$\int 5 - 4x^2 dx = 5x - 4 \cdot \frac{x^3}{3} + C = \underline{5x - \frac{4}{3}x^3 + C}$$

$$\int \frac{1-x^2}{x} dx = \int \frac{1}{x} - x \; dx = \underline{\ln|x| - \frac{x^2}{2} + C}$$

$$\int x \cdot \ln(x) \, dx = \; ?$$ integration by parts

$$\int \frac{x}{1-x^2} dx = \; ?$$ substitution

$$\int \frac{1}{1-x^2} dx = \; ?$$ partial fractions

$$\left(\int e^{-x^2} dx = \; ? \right.$$ no answer among "elementary functions")

Integration techniques:

a) Integration by parts:

$$\int u'\cdot v\,dx = u\cdot v - \int u\cdot v'\,dx$$

Motivation:

$$(u\cdot v)' = u'v + uv'$$

$$u\cdot v = \underline{\int u'v\,dx} + \int uv'\,dx$$

Ex: $\int x\cdot e^x\,dx =$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= xe^x - \int e^x\,dx$$

$$= \underline{\underline{xe^x - e^x + C}}$$

$$\xcancel{\boxed{\begin{array}{ll} u = x^2/2 & v = e^x \\ u' = x & v' = e^x \end{array}} \quad \int x\cdot e^x\,dx = \frac{x^2}{2}\cdot e^x - \int \frac{x^2}{2}\cdot e^x\,dx}$$

Ex: $\int \ln x\,dx = \int 1\cdot \ln x\,dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x\cdot \ln x - \int x\cdot \frac{1}{x}\,dx$$

$$= x\cdot \ln x - \int 1\,dx = \underline{\underline{x\ln x - x + C}}$$

b) Substitution:

Ex:

$$\int \frac{x}{1-x^2}\,\boxed{dx} = \int \frac{\cancel{x}}{u}\;\frac{du}{\boxed{-2\cancel{x}}}$$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= u'\cdot dx \end{aligned}}$$

$$u' = -2x$$

$$du = -2x\cdot dx$$

$$\Downarrow$$

$$dx = \frac{du}{-2x}$$

$$= \int \underbrace{\frac{1}{u}\cdot\frac{1}{-2}}_{\text{expr. in } u}\,du$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|1-x^2| + C$$

Ex:

$$\int 2x\,e^{-x^2}\,dx = \int \cancel{2x}\,e^{u}\cdot\frac{du}{-\cancel{2x}}$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$= \int -e^{u}\,du = -e^{u} + C$$

$$= -e^{-x^2} + C$$

c) Partial fractions:

No time today,
will be covered Friday.

Problems for Lecture 4

1. Compute the integrals

a) $\int x^2 \ln x \, dx$

b) $\int \frac{x^3 + 2x^2 + 1}{x} \, dx$

c) $\int x e^{-x^2} dx$

d) $\int \frac{x}{x^2+1} \, dx$

e) $\int \frac{1}{(2x-3)^2} dx$

f) $\int x^3 e^{-x^2} dx$

g) $\int e^{\sqrt{x}} \, dx$

2 Simplify the expressions using polynomial division:

a) $\frac{x^3}{x^2-1}$

b) $\frac{x^2+2x-3}{x+1}$

c) $\frac{x^3+x^2+x+1}{x+1}$

d) $\frac{x^4+1}{x-1}$

3. Compute the integrals:

a) $\int \frac{3}{2x-4} dx$

b) $\int \frac{1}{x^2+x} dx$

c) $\int \frac{x}{x^2-4x-5} dx$

d) $\int \frac{x^2}{x^2-1} dx$

e) $\int \frac{1}{x^2-4x+4} dx$

Solutions for Lecture 4.

1. a) $\int \underset{\substack{\shortparallel \\ v'}}{x^2}\ \underset{\substack{\shortparallel \\ u}}{\ln x}\,dx = \underset{\substack{\shortparallel \\ v}}{\tfrac{1}{3}x^3}\cdot \underset{\substack{\shortparallel \\ u}}{\ln x} - \int \underset{\substack{\shortparallel \\ v}}{\tfrac{1}{3}x^3}\cdot \underset{\substack{\shortparallel \\ u'}}{\tfrac{1}{x}}\,dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx = \underline{\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C}$

b) $\int \frac{x^3+2x^2+1}{x}dx = \int x^2+2x+\frac{1}{x}\,dx = \underline{\frac{1}{3}x^3+x^2+\ln|x|+C}$

c) $\int x e^{-x^2}dx = \int \cancel{x} e^{u}\cdot\frac{du}{-2\cancel{x}} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C$

$\boxed{\begin{array}{l} u=-x^2 \\ du=-2x\,dx \end{array}}$

$= \underline{-\frac{1}{2}e^{-x^2}+C}$

d) $\int \frac{x}{x^2+1}dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u|+C$

$\boxed{\begin{array}{l} u=x^2+1 \\ du=2x\,dx \end{array}}$

$= \frac{1}{2}\ln|x^2+1|+C$

$= \underline{\frac{1}{2}\ln(x^2+1)+C}$

e) $\int \frac{1}{(2x-3)^2}dx = \int\frac{1}{u^2}\frac{du}{2} = \frac{1}{2}\int u^{-2}du$

$\boxed{\begin{array}{l} u=2x-3 \\ du=2\,dx \end{array}}$

$= \frac{1}{2}\left(-\frac{1}{u}\right)+C$

$= \underline{-\frac{1}{2}\cdot\frac{1}{2x-3}+C}$

f) $\int x^3 e^{-x^2}dx = \int x^3 e^u \frac{du}{-2x} = \int -\frac{1}{2}x^2 e^u du$

$\boxed{\begin{array}{l} u=-x^2 \\ du=-2x\,dx \end{array}}$

$= -\frac{1}{2}\int(-u)e^u du$

$x^2=-u$

$= \frac{1}{2}\int ue^u du \underset{\substack{\uparrow \\ \text{int. by} \\ \text{parts}}}{=} ue^u - \int 1\cdot e^u du = ue^u - e^u + C$

$= \underline{-x^2e^{-x^2} - e^{-x^2} + C}$

9) $\int e^{\sqrt{x}}\,dx = \int e^{u} \cdot 2\sqrt{x}\,du$

$$\left\{ \begin{array}{l} u=\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array} \right\}$$

$$= \int e^{u} \cdot 2u\,du$$

$\sqrt{x}=u$

$$= \int 2ue^{u}\,du \underset{\text{int. by parts}}{=} 2ue^{u} - \int 2e^{u}\,du$$

$$= 2ue^{u} - 2e^{u} + C = \underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$$

$\underline{2}$ a) $x^3 : x^2-1 = x \Rightarrow \underline{\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}}$

$-(x^3-x)$

x

b) $x^2+2x-3 : x+1 = x+1 \Rightarrow \frac{x^2+2x-3}{x+1} = \underline{x+1+\frac{-4}{x+1}}$

$-(x^2+x\ \ )$

$x-3$

$-(x+1)$

-4

c) $x^3+x^2+x+1 : x+1 = x^2+1 \Rightarrow \frac{x^3+x^2+x+1}{x+1} = \underline{x^2+1}$

$-(x^3+x^2\ \ )$

$x+1$

$x+1$

0

d) $x^4+1 : x-1 = x^3+x^2+x+1 \Rightarrow \frac{x^4+1}{x-1} = \underline{x^3+x^2+x+1+\frac{2}{x-1}}$

$-(x^4-x^3)$

x^3+1

$-(x^3-x^2)$

x^2+1

$-(x^2-x)$

$x+1$

$-(x-1)$

2

3. a) $\int \frac{3}{2x-4}\,dx = \int \frac{3}{u}\frac{du}{2} = \frac{3}{2}\int \frac{1}{u}\,du$

$\left(\begin{array}{l} u = 2x-4 \\ du = 2dx \end{array} \right)$ $= \frac{3}{2}\ln|2x-4| + C$

b) $\int \frac{1}{x^2+x}\,dx \underset{\uparrow}{=} \int \frac{1}{x} + \frac{-1}{x+1}\,dx = \ln|x| - \ln|x+1| + C$

$\frac{1}{x^2+x} = \frac{1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big| \cdot (x+1)x$

$1 = A\cdot(x+1) + Bx$

$= (\underset{0}{\underset{\shortparallel}{A+B}})x + (\underset{1}{\underset{\shortparallel}{A}})$

$A = 1, \quad B = -1$

$\overset{\uparrow}{=} \ln\left|\frac{x}{x+1}\right| + C$

rules for logarithms

c) $\int \frac{x}{x^2-4x-5}\,dx \underset{\uparrow}{=} \int \frac{5/6}{x-5} + \frac{1/6}{x+1}\,dx$

$x^2-4x-5 = (x-5)(x+1)$

$\frac{x}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1} \quad \Big| \cdot (x-5)(x+1)$

$x = A(x+1) + B(x-5)$

$= (\underset{1}{\underset{\shortparallel}{A+B}})x + (\underset{0}{\underset{\shortparallel}{A-5B}})$

$A+B = 1$

$A-5B = 0 \rightarrow A = 5B$

$5B + B = 1$

$6B = 1$

$A = 5/6 \quad B = 1/6$

$= \frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C$

$= \frac{1}{6}\left(\ln|x-5|^5 + \ln|x+1|\right) + C$

$= \frac{1}{6}\ln|(x-5)^5(x+1)| + C$

d) $\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = x + \int \frac{1}{(x-1)(x+1)}\,dx$

$$\begin{array}{l} x^2 : x^2-1 = 1 \\ \underline{-(x^2-1)} \\ \quad 1 \end{array}$$

$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad | \cdot (x-1)(x+1)$

$1 = A(x+1) + B(x-1)$

$= \underset{\substack{\shortparallel \\ 0}}{(A+B)}x + \underset{\substack{\shortparallel \\ 1}}{(A-B)}$

$$\begin{array}{l} A+B=0 \\ \underline{A-B=1} \\ 2A \quad = 1 \end{array} \longrightarrow \underline{A=1/2}, \ \underline{B=-1/2}$$

$= x + \int \frac{1/2}{x-1} + \frac{-1/2}{x+1}\,dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

e) $\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$

$u = x-2$
$du = 1 \cdot dx$

$= -\frac{1}{x-2} + C$