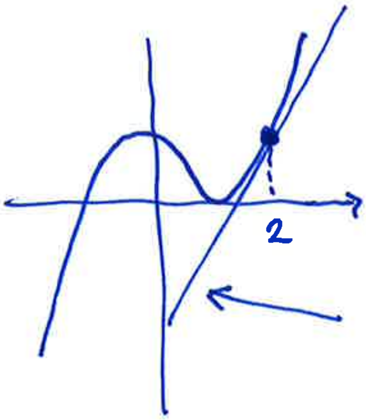
Plan:

1. Functions and derivatives
2. Exponential functions and logarithms
3. Higher derivatives

Reading:

[ME] Ch. 2-5

1 Functions and derivatives



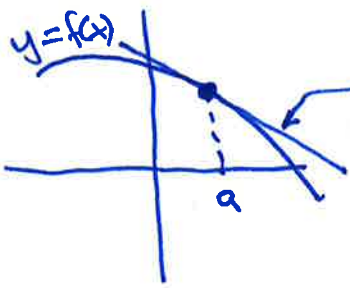
Ex: $f(x) = x^3 - 3x + 2$

$f(2) = 2^3 - 3\cdot 2 + 2 = 4$

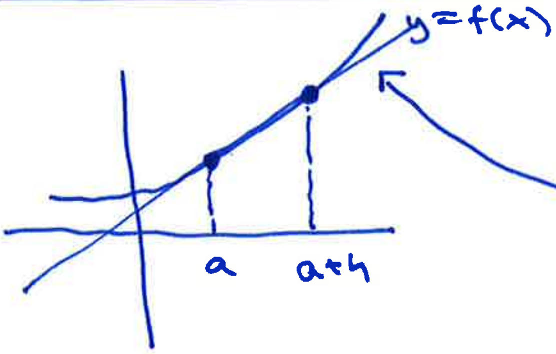
the tangent line through $(2,4)$ (of f)

= the straight line through $(2,4)$ that approximates f best close to $x=2$

Definition: $f'(a)$ = the slope of the tangent line of f through the point $(a, f(a))$



= slope of



- Want to compute $f'(a)$
- choose h (not too big)

Secant line through

$(x_1, y_1) \longrightarrow (a, f(a))$

and

$(x_2, y_2) \longrightarrow (a+h, f(a+h))$

Defn:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

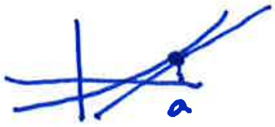
Slope of secant line:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(\cancel{a}+h) - \cancel{a}}$$

$$= \frac{f(a+h) - f(a)}{h}$$

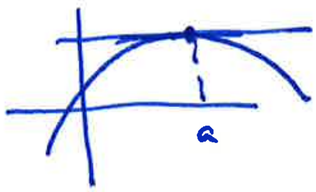
Interpretation:

$f'(a) > 0$

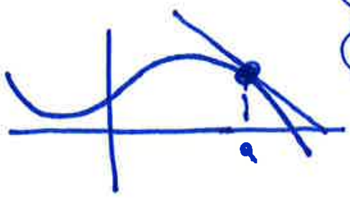


(graph increasing close to $x=a$)

$f'(a) = 0$



$f'(a) < 0$



(graph decreasing close to $x=a$)

Computing derivatives:

- directly from the definition
- derivation rules

Derivative as a function: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

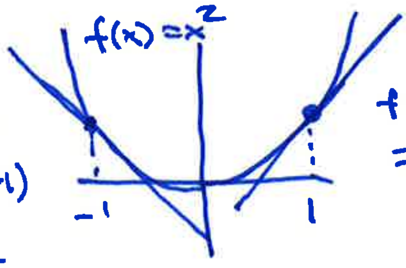
Derivation rules:

i) $(x^n)' = n x^{n-1}$ for all numbers n } when $f(x) = x^n$, then $f'(x) = n \cdot x^{n-1}$

ii) $\left(u(x) \pm v(x) \right)' = u'(x) \pm v'(x)$

iii) $\left(c \cdot u(x) \right)' = c \cdot u'(x)$ for all constants c

$f'(-1) = 2 \cdot (-1) = \underline{\underline{-2}}$



$f'(1) = 2 \cdot 1 = \underline{\underline{2}}$

Ex: $(x^2)' = \underline{2x}$

$f(x) = x^2 \Rightarrow f'(x) = 2x$

$$(3x^2 - 1)' = (3x^2)' - (1)'$$
$$= 3 \cdot 2x - 0 = \underline{\underline{6x}}$$

$$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2} \cdot x^{\frac{1}{2} - 1} = \frac{1}{2} \cdot x^{-1/2}$$
$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \underline{\underline{\frac{1}{2\sqrt{x}}}}$$

$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = (-2)\cdot x^{-2-1} = -2x^{-3}$$
$$= -\frac{2}{x^3}$$

iv) $(u(x)\cdot v(x))' = u'(x)\cdot v(x) + u(x)\cdot v'(x)$

or $(u\cdot v)' = u'v + u\cdot v'$

v) $\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{v(x)^2}$

or $\left(\frac{u}{v}\right)' = \frac{u'\cdot v - uv'}{v^2}$

Ex: $\left(\frac{x}{x-1}\right)' = \frac{(x)'\cdot(x-1) - x\cdot(x-1)'}{(x-1)^2}$

$$= \frac{1\cdot(x-1) - x\cdot 1}{(x-1)^2} = \frac{\cancel{x}-1-\cancel{x}}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$\left(\frac{x-1}{x}\right)' = \left(1-\frac{1}{x}\right)' = \left(-x^{-1}\right) = -\left(-1\cdot x^{-2}\right)$$
$$= 1/x^2$$

vi) Chain rule:

Ex: $f(x) = (2x-1)^5 = \underline{u^5}$, where $u = \underline{2x-1}$

$f'(x) = 5u^4 \cdot u'(x)$

$= 5u^4 \cdot 2 = 10u^4 = \underline{\underline{10 \cdot (2x-1)^4}}$

Chain rule: $f(x) = f(u(x)) \Rightarrow f'(x) = f'(u) \cdot u'(x)$

$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}$$

Notation: $f'(x) = (f(x))' = \frac{df}{dx}$

$$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$$

Ex: $f(x) = \sqrt{x^2+1} = \sqrt{u}$, $u = \underline{x^2+1}$

$f'(x) = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \underline{\underline{\frac{x}{\sqrt{x^2+1}}}}$

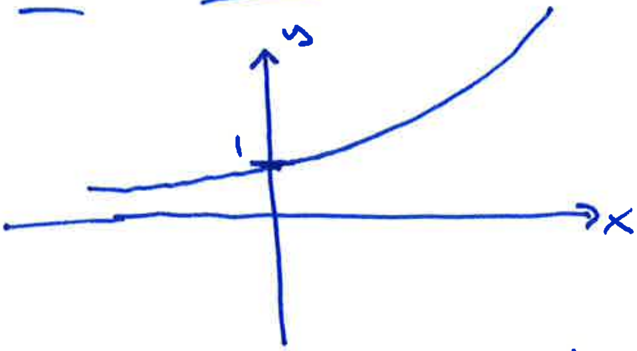
② Logarithms and exponential functions

Exponential function:

$f(x) = a^x$ for a given number $\underline{a > 0}$

Ex: $a = 1{,}04$ $\qquad f(x) = 1.04^x$

$(1{,}04^x)' = 1{,}04^x \cdot \ln(1{,}04)$



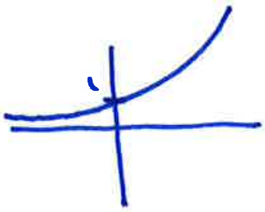
$f(-3) = 1{,}04^{-3} = \dfrac{1}{1{,}04^3}$

$$\boxed{\begin{array}{l}\text{Derivative of exponential functions:}\\ \text{viii)}\quad (a^x)' = a^x \cdot \ln(a)\end{array}}$$

Properties of exponential functions: $\boxed{f(x) = a^x, \quad a > 0}$

i) f is defined for all x

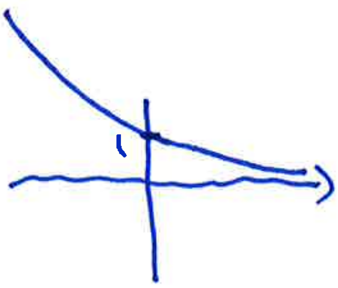
ii) $\underline{a > 1}$:



$f(x) > 0$ for all x
$f(0) = 1$
$f(x) \to 0$ as $x \to -\infty$
$f(x) \to \infty$ " $x \to \infty$
f increasing with $f' = a^x \cdot \ln(a)$

or

$\underline{0 < a < 1}$:



$f(x) > 0$ for all x
$f(0) = 1$
$f(x) \to \infty$ as $x \to -\infty$
$f(x) \to 0$ " $x \to \infty$
f decreasing with $f' = a^x \cdot \ln(a)$

Ex: $f(x) = 0.96^x$

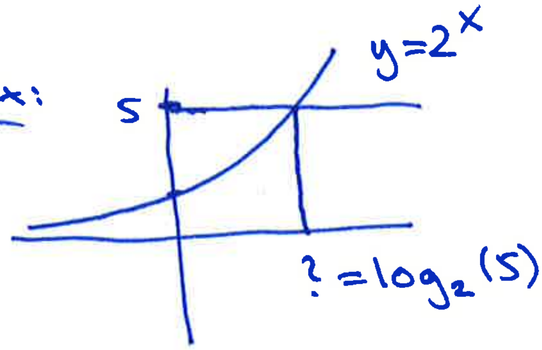
iii) Rules for powers with base a

$$a^{m+n} = a^m \cdot a^n \qquad (a^m)^n = a^{m \cdot n}$$

$$a^{m-n} = a^m / a^n$$

Logarithms: $f(x) = \log_a(x)$ for $a>0$

Ex:



$$2^x = 5$$

$$\log_2(2^x) = \log_2(5)$$

$$x = \log_2(5)$$

$\log_a(x)$ = the solution u of the equation

$$a^u = x$$

$\log_a(x)$ is the inverse function of a^x

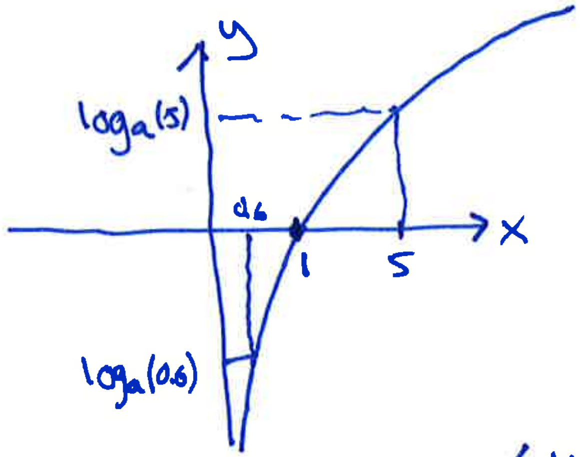
Ex: $\log_2(4) = 2$, $\log_2(8) = 3$ $\Rightarrow$ $\log_2(5)$ is a number between 2 and 3, can compute an approximation using a calculator

Properties of logarithms:

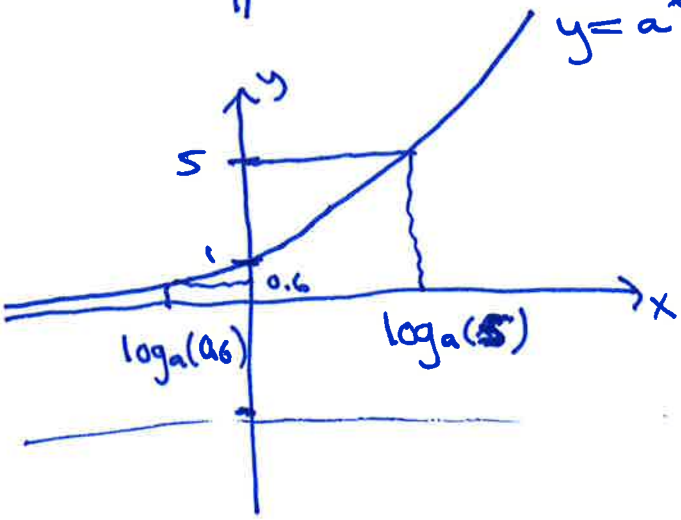
i) $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

ii) $\log_a(x/y) = \log_a(x) - \log_a(y)$

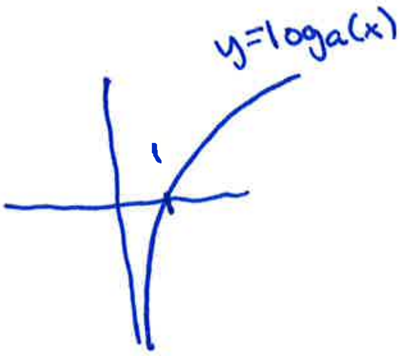
iii) $\log_a(x^y) = y \cdot \log_a(x)$



$y = \log_a(x)$ when $\underline{a > 1}$



$y = a^x$ when $\underline{a > 1}$



Summary: $f(x) = \log_a(x)$

i) defined when $\underline{x > 0}$

ii) $\log_a(1) = 0$

iii) If $a > 1$: $\log_a(x) > 0$ for $x > 1$

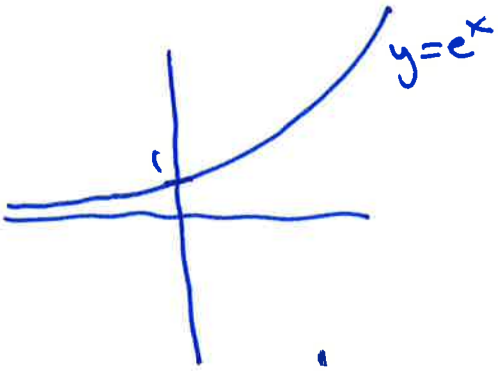
$\log_a(x) < 0$ " $x < 1$

iv) $\log_a(x) \to \infty$ as $x \to \infty$ and $\log_a(x) \to -\infty$ as $x \to 0$ $(a > 1)$

Important special case: $a = e = 2.71828\ldots$

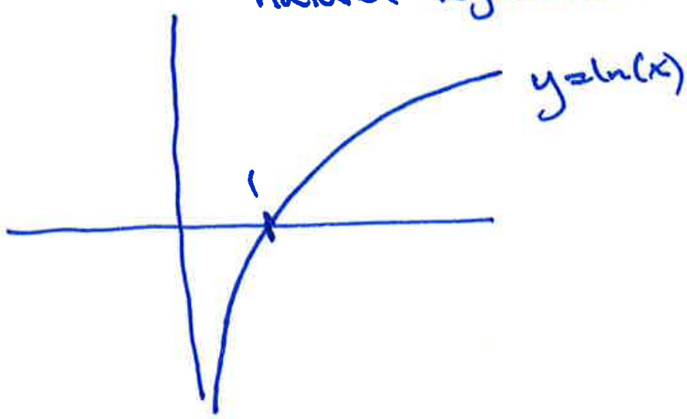
(Euler's number)

$f(x) = e^x$



$f(x) = \ln(x)$ $\left(= \log_e(x)\right)$

natural logarithm



$$\boxed{\textbf{Formula:} \quad \log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

Ex: $2^x = 5$

$\log_2(2^x) = \log_2(5)$

$x = \log_2(5)$

$\ln(2^x) = \ln(5)$

$\frac{x \cdot \cancel{\ln(2)}}{\cancel{\ln(2)}} = \frac{\ln(5)}{\ln(2)}$

$x = \frac{\ln(5)}{\ln(2)}$

Derivatives:

$$(a^x)' = a^x \cdot \ln(a)$$

$$(e^x)' = e^x$$

$$(\ln x)' = 1/x$$

$$(\log_a x)' = \frac{1}{x} \cdot \frac{1}{\ln(a)}$$

Ex:

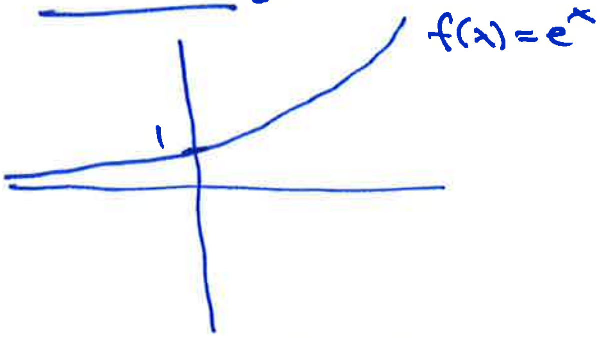
$$(2^x)' = 2^x \cdot \ln(2)$$
$$= 2^x \cdot 0{,}693\ldots$$

$$(1{,}04^x)' = 1{,}04^x \cdot \ln(1{,}04)$$

$$(e^x)' = e^x \cdot \ln(e)$$
$$= e^x \cdot 1 = e^x$$

$$\ln(e) = \ln(e^1) = 1$$

Summary:

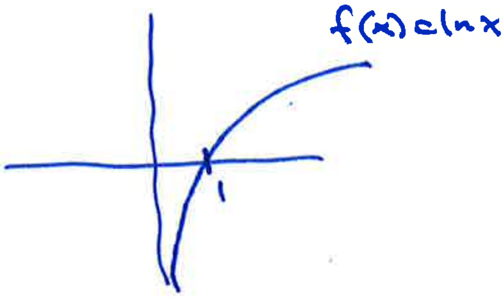


$D_f = \mathbb{R}$ (defined for all x)

$V_f = (0, \to)$

f is increasing

$f'(x) = e^x$



$D_f = (0, \to)$

$V_f = \mathbb{R}$ (all y-values)

f is increasing

$f'(x) = 1/x$

Ex: $(e^{2-x})' = e^{2-x} \cdot (-1) = \underline{\underline{-e^{2-x}}}$

$(x \cdot \ln x)' = (x)' \cdot \ln x + x \cdot (\ln x)'$

$= 1 \cdot \ln x + x \cdot \frac{1}{x} = \underline{\underline{\ln(x) + 1}}$

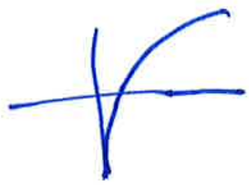
③ **Higher derivatives:**

$f''(x) = (f'(x))'$

Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = \underline{3x^2 - 3}$

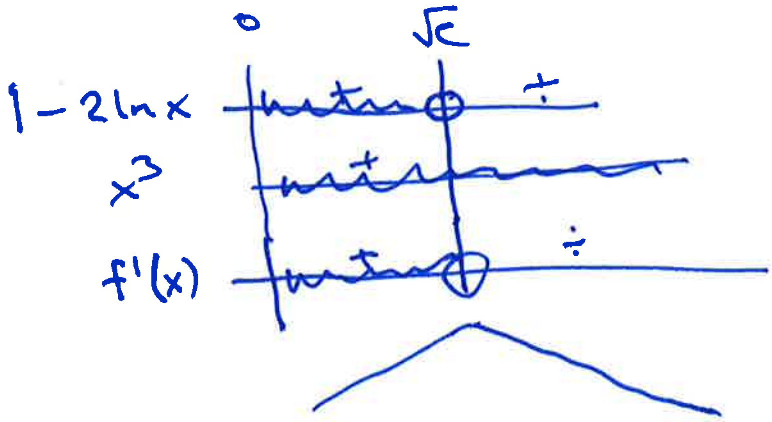
$f''(x) = 3 \cdot 2x = \underline{6x}$



Ex: $f(x) = \frac{\ln x}{x^2}$, $x > 0$

$f'(x) = \frac{(1/x) \cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x \cdot \ln(x)}{x^4}$

$= \frac{\cancel{x} \cdot (1 - 2\ln x)}{x^{\cancel{4}\,3}} = \underline{\underline{\frac{1 - 2\ln x}{x^3}}}$



$1 - 2\ln x = 0$

$\ln x = 1/2$

$e^{\ln x} = e^{1/2}$

$x = \sqrt{e}$

EXERCISES

2.10 *a*) Use the geometric definition of the derivative to prove that the derivative of a constant function is 0 everywhere and the derivative of $f(x) = mx$ is $f'(x) = m$ for all x.

b) Use the method of the proof of Theorem 2.2 to prove that the derivative of x^3 is $3x^2$ and the derivative of x^4 is $4x^3$.

2.11 Find the derivative of the following functions at an arbitrary point:

a) $-7x^3$,

b) $12x^{-2}$,

c) $3x^{-3/2}$,

d) $\frac{1}{2}\sqrt{x}$,

e) $3x^2 - 9x + 7x^{2/5} - 3x^{1/2}$,

f) $4x^5 - 3x^{1/2}$,

g) $(x^2 + 1)(x^2 + 3x + 2)$,

h) $(x^{1/2} + x^{-1/2})(4x^5 - 3\sqrt{x})$,

i) $\dfrac{x-1}{x+1}$,

j) $\dfrac{x}{x^2+1}$,

k) $(x^5 - 3x^2)^7$,

l) $5(x^5 - 6x^2 + 3x)^{2/3}$,

m) $(x^3 + 2x)^3(4x + 5)^2$.

2.12 Find the equation of the tangent line to the graph of the given function for the specified value of x. [Hint: Given a point on a line and the slope of the line, one can construct the equation of the line.]

a) $f(x) = x^2$, $x_0 = 3$; *b*) $f(x) = x/(x^2 + 2)$, $x_0 = 1$.

2.13 Prove parts *a* and *b* of Theorem 2.4.

2.14 In Theorem 2.3, we proved that the derivative of $y = x^k$ is $y' = kx^{k-1}$ for all positive integers k. Use the Quotient Rule, Theorem 2.4*d*, to extend this result to negative integers k.

2.5 DIFFERENTIABILITY AND CONTINUITY

As we saw in Section 2.3, a function f is differentiable at x_0 if, geometrically speaking, its graph has a tangent line at $(x_0, f(x_0))$, or analytically speaking, the limit

$$\lim_{h_n \to 0} \frac{f(x_0 + h_n) - f(x_0)}{h_n} \tag{8}$$

exists and is the same for every sequence $\{h_n\}$ which converges to 0. If a function is differentiable at every point x_0 in its domain D, we say that the function is **differentiable**. Only functions whose graphs are "smooth curves" have tangent lines everywhere; in fact, mathematicians commonly use the word "smooth" in place of the word "differentiable."

Negative bases are not allowed for the exponential function. For example, the function $k(x) = (-2)^x$ would take on positive values for x an even integer and negative values for x an odd integer; yet it is never zero in between. Furthermore, since you cannot take the square root of a negative number, the function $(-2)^x$ is not even defined for $x = 1/2$ or, more generally, whenever x is a fraction p/q and q is an even integer. So, we can only work with exponential functions a^x, where a is a number greater than 0.

EXERCISES

5.1 Evaluate each of the following:

$$2^3, \quad 2^{-3}, \quad 8^{1/3}, \quad 8^{2/3}, \quad 8^{-2/3}, \quad \pi^0, \quad 64^{-5/6}, \quad 625^{3/4}, \quad 25^{-5/2}.$$

5.2 Sketch the graph of: *a)* $y = 5^x$; *b)* $y = .2^x$; *c)* $y = 3(5^x)$; *d)* $y = 1^x$.

5.2 THE NUMBER *e*

Figure 5.2 presented graphs of exponential functions with bases 2, 3, and 10, respectively. We now introduce a number which is the most important base for an exponential function, the irrational number e. To motivate the definition of e, consider the most basic economic situation — the growth of the investment in a savings account. Suppose that at the beginning of the year, we deposit \$$A$ into a savings account which pays interest at a simple annual interest rate r. If we will let the account grow without deposits or withdrawals, after one year the account will grow to $A + rA = A(1 + r)$ dollars. Similarly, the amount in the account in any one year is $(1 + r)$ times the previous year's amount. After two years, there will be

$$A(1 + r)(1 + r) = A(1 + r)^2$$

dollars in the account. After t years, there will be $A(1 + r)^t$ dollars in the account.

Next, suppose that the bank compounds interest four times a year; at the end of each quarter, it pays interest at $r/4$ times the current principal. After one quarter of a year, the account contains $A + \frac{r}{4}A$ dollars. After one year, that is, after four compoundings, there will be $A\left(1 + \frac{r}{4}\right)^4$ dollars in the account. After t years, the account will grow to $A\left(1 + \frac{r}{4}\right)^{4t}$ dollars.

More generally, if interest is compounded n times a year, there will be $A\left(1 + \frac{r}{n}\right)$ dollars in the account after the first compounding period, $A\left(1 + \frac{r}{n}\right)^n$ dollars in the account after the first year, and $A\left(1 + \frac{r}{n}\right)^{nt}$ dollars in the account after t years.

Many banks compound interest daily; others advertise that they compound interest *continuously*. By what factor does money in the bank grow in one year at

Base e Logarithms

Since *the* exponential function $\exp(x) = e^x$ has all the properties that 10^x has, it also has an inverse. Its inverse works the same way that $\operatorname{Log} x$ does. Mirroring the fundamental role that e plays in applications, the inverse of e^x is called the **natural logarithm** function and is written as $\ln x$. Formally,

$$\ln x = y \iff e^y = x;$$

$\ln x$ is the power to which one must raise e to get x. As we saw in general in (3), this definition can also be summarized by the equations

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x. \tag{4}$$

The graph of e^x and its reflection across the diagonal, the graph of $\ln x$, are similar to the graphs of 10^x and $\operatorname{Log} x$ in Figure 5.4.

Example 5.2 Let's work out some examples. The natural log of 10 is the power of e that gives 10. Since e is a little less than 3 and $3^2 = 9$, e^2 will be a bit less than 9. We have to raise e to a power bigger than 2 to obtain 10. Since $3^3 = 27$, e^3 will be a little less than 27. Thus, we would expect that $\ln 10$ to lie between 2 and 3 and somewhat closer to 2. Using a calculator, we find that the answer to four decimal places is $\ln 10 = 2.3026$.

We list a few more examples. Cover the right-hand side of this table and try to estimate these natural logarithms.

$$\begin{array}{llll}
\ln e & = 1 & \text{since } e^1 & = e; \\
\ln 1 & = 0 & \text{since } e^0 & = 1; \\
\ln 0.1 & = -2.3025\cdots & \text{since } e^{-2.3025\cdots} & = 0.1; \\
\ln 40 & = 3.688\cdots & \text{since } e^{3.688\cdots} & = 40; \\
\ln 2 & = 0.6931\cdots & \text{since } e^{0.6931\cdots} & = 2.
\end{array}$$

EXERCISES

5.3 First estimate the following logarithms without a calculator. Then, use your calculator to compute an answer correct to four decimal places:

a) $\operatorname{Log} 500$, *b*) $\operatorname{Log} 5$, *c*) $\operatorname{Log} 1234$, *d*) $\operatorname{Log} e$,
e) $\ln 30$, *f*) $\ln 100$, *g*) $\ln 3$, *h*) $\ln \pi$.

EXERCISES

5.5 Solve the following equations for x:

$$a)\ 2e^{6x} = 18; \quad b)\ e^{x^2} = 1; \quad c)\ 2^x = e^5;$$

$$d)\ 2^{x-2} = 5; \quad e)\ \ln x^2 = 5; \quad f)\ \ln x^{5/2} - 0.5 \ln x = \ln 25.$$

5.6 Derive a formula for the amount of time that it takes money to triple in a bank account that pays interest at rate r compounded continuously.

5.7 How quickly will \$500 grow to \$600 if the interest rate is 5 percent compounded continuously?

5.5 DERIVATIVES OF EXP AND LOG

To work effectively with exponential and logarithmic functions, we need to compute and use their derivatives. The natural logarithmic and exponential functions have particularly simple derivatives, as the statement of the following theorem indicates.

Theorem 5.2 The functions e^x and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order. Their first derivatives are given by

$$a)\quad (e^x)' = e^x,$$

$$b)\quad (\ln x)' = \frac{1}{x}.$$

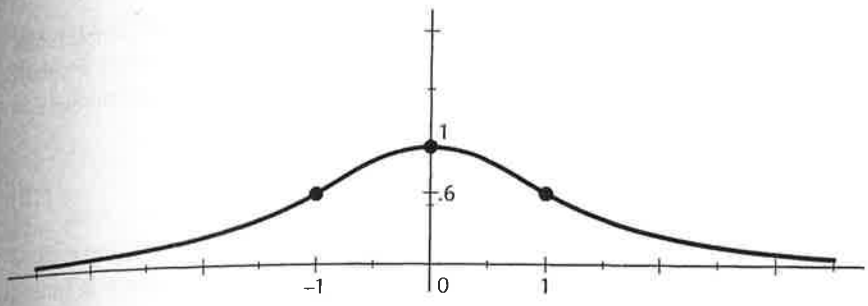
If $u(x)$ is a differentiable function, then

$$c)\quad \left(e^{u(x)}\right)' = \left(e^{u(x)}\right) \cdot u'(x),$$

$$d)\quad (\ln u(x))' = \frac{u'(x)}{u(x)} \qquad \text{if } u(x) > 0.$$

We will prove this theorem in stages. That the exponential map is continuous should be intuitively clear from the graph in Figure 5.4; its graph has no jumps or discontinuities. Since the graph of $\ln x$ is just the reflection of the graph of e^x across the diagonal $\{x = y\}$, the graph of $\ln x$ has no discontinuities either, and so the function $\ln x$ is continuous for all x in the set $\mathbf{R}_{++}$ of positive numbers.

It turns out to be easier to compute the derivative of the natural logarithm first.



The graph of $e^{-x^2/2}$.

Figure 5.5

Proof Since $b = e^{\ln b}$, then $b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$. By equation b in Example 5.5,

$$(b^x)' = \left(e^{(\ln b)\cdot x}\right)' = (\ln b)(e^{(\ln b)\cdot x}) = (\ln b)(b^x). \quad \blacksquare$$

Example 5.7 $(10^x)' = (\ln 10)(10^x)$.

Note that $(b^x)' = b^x$ if and only if $\ln b = 1$, that is, if and only if $b = e$. In fact, *the* exponential functions $y = ke^x$ are the only functions which are equal to their derivatives throughout their domains. This fact gives another justification for e being considered the *natural* base for exponential functions.

EXERCISES

5.8 Compute the first and second derivatives of each of the following functions:
a) xe^{3x}, *b)* e^{x^2+3x-2}, *c)* $\ln(x^4+2)^2$, *d)* $\dfrac{x}{e^x}$, *e)* $\dfrac{x}{\ln x}$, *f)* $\dfrac{\ln x}{x}$.

5.9 Use calculus to sketch the graph of each of the following functions:
a) xe^x, *b)* xe^{-x}, *c)* $\cosh(x) \equiv (e^x + e^{-x})/2$.

5.10 Use the equation $10^{\text{Log}\,x} = x$, Example 5.7, and the method of the proof of Lemma 5.3 to derive a formula for the derivative of $y = \text{Log}\,x$.

5.6 APPLICATIONS

Present Value

Many economic problems entail comparing amounts of money at different points of time in the same computation. For example, the benefit/cost analysis of the construction of a dam must compare in the same equation this year's cost of construction, future years' costs of maintaining the dam, and future years' monetary

Problems for Lecture 2

1. Compute $f'(x)$ and $f''(x)$.

a) $f(x) = x^4 - x\sqrt{x} + 3x$, $x > 0$

b) $f(x) = x\sqrt{x^2+1}$

c) $f(x) = \dfrac{x+1}{x^2-3x+2}$, $x \neq 1, 2$

d) $f(x) = xe^x - x^2e^{-x} + e^{2x-1}$

e) $f(x) = \ln(x) - \ln(x-1)$, $x > 1$

f) $f(x) = \ln(x^3 - x^2)$, $x > 1$

2. Is f convex, concave, or neither?

a) $f(x) = x^2$

b) $f(x) = x^a$ $(a > 0)$

c) $f(x) = e^x$

d) $f(x) = \ln(x)$, $x > 0$

e) $f(x) = \ln(x^3 - x^2)$, $x > 1$

Solutions for Lecture 2

1.

a) $f'(x) = 4x^3 - \frac{3}{2}\sqrt{x} + 3$

$$f''(x) = 12x^2 - \frac{3}{4\sqrt{x}}$$

b) $f'(x) = 1 \cdot \sqrt{x^2+1} + x \cdot \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{\sqrt{x^2+1}\cdot\sqrt{x^2+1} + x^2}{\sqrt{x^2+1}}$

$$= \frac{2x^2+1}{\sqrt{x^2+1}}$$

$$f''(x) = \frac{4x\cdot\sqrt{x^2+1} - (2x^2+1)\cdot\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot\cancel{2}x}{x^2+1}$$

$$= \frac{4x(x^2+1) - x(2x^2+1)}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{2x^3+3x}{(x^2+1)^{3/2}}$$

c) $f'(x) = \left(\frac{x+1}{x^2-3x+2}\right)' = \frac{1\cdot(x^2-3x+2) - (x+1)\cdot(2x-3)}{(x^2-3x+2)^2}$

$$= \frac{(x^2-3x+2) - (2x^2-x-3)}{(x^2-3x+2)^2} = \frac{-x^2-2x+5}{(x^2-3x+2)^2}$$

$$f''(x) = \frac{(-2x-2)(x^2-3x+2)^2 - (-x^2-2x+5)\cdot 2(x^2-3x+2)\cdot(2x-3)}{(x^2-3x+2)^4}$$

$$= \frac{(-2x-2)(x^2-3x+2) - (4x-6)(-x^2-2x+5)}{(x^2-3x+2)^3}$$

$$= \frac{2x^3+6x^2-30x+26}{(x^2-3x+2)^3}$$

d) $f'(x) = 1\cdot e^x + \cancel{x}e^x - 2xe^{-x} - x^2e^{-x}\cdot(-1) + e^{2x-1}\cdot 2$

$= (x+1)e^x + (x^2-2x)e^{-x} + 2e^{2x-1}$

$f''(x) = (x+2)e^x + (2x-2)e^{-x} + (x^2-2x)e^{-x}\cdot(-1) + 4e^{2x-1}$

$= (x+2)e^x + (-x^2+4x-2)e^{-x} + 4e^{2x-1}$

e) $f(x) = \ln(x) - \ln(x-1)$

$f'(x) = \frac{1}{x} - \frac{1}{x-1} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}$

$f''(x) = -\frac{1}{x^2} - \left(-\frac{1}{(x-1)^2}\right)\cdot(+1) = -\frac{1}{x^2} + \frac{1}{(x-1)^2} = \frac{-(x-1)^2+x^2}{x^2(x-1)^2}$

$= \underline{\underline{\frac{2x-1}{x^2(x-1)^2}}}$

f) $f'(x) = \frac{1}{x^3-x^2}\cdot(3x^2-2x) = \frac{3x^2-2x}{x^2(x-1)} = \frac{x(3x-2)}{x^2(x-1)} = \frac{3x-2}{x(x-1)}$

$f''(x) = \left(\frac{3x-2}{x^2-x}\right)' = \frac{3(x^2-x)-(3x-2)\cdot(2x-1)}{(x^2-x)^2} = \frac{-3x^2+4x-2}{x^2(x-1)^2}$

2\. a) $f'(x) = 2x \quad f''(x) = 2 > 0$ for all $x \rightarrow$ convex

b) $f'(x) = ax^{a-1} \quad f''(x) = a\cdot(a-1)x^{a-2}$

since $a>0$, $x^{a-2}>0$,
we have that

$f''(x) > 0$ when $a>1$ $\rightarrow$ $a>1$: f convex
$f''(x) < 0$ " $a<1$ $\quad 0<a<1$: f concave

c) $f' = f'' = e^x > 0 \rightarrow$ convex

d) $f' = \ln x \quad f'' = \frac{1}{x} = x^{-1} \quad f''' = -\frac{1}{x^2} < 0 \rightarrow f$ concave

e) $f'' = \frac{-3x^2+4x-2}{x^2(x-1)^2}$ by (f), and $x^2, (x-1)^2 > 0$.

$-3x^2+4x-2=0$

$x = \frac{-4\pm\sqrt{16-4\cdot 6}}{2\cdot(-3)}$ no real solution $\rightarrow$ no intersection with x-axis $\rightarrow$ f concave

$f''(x) < 0$ for all $x>1$