Plan:

① Introduction

② Solving linear systems

③ Gaussian elimination

Reading:

[ME] 6.1, 7.1-7.3

([ME] = Mathematics for Economists, by Simon & Blume)

① Introduction

Mathematics: Wed-Fri this week

Economics, econometrics: next week

{ 09-12: Matrices & Vectors
{ 14-17: Calculus

② Solving linear systems

Linear system:

Ex:

$$\begin{aligned} 7x - 3y &= 6 \\ 2x + y &= 11 \end{aligned}$$

2x2 linear system

(2 equations
2 variables
equations are linear)

Ex:

$$\begin{aligned} x - y - 2z + 3w &= 2 \\ 2x - y + z + 5w &= 3 \\ x + 7y + 10z + 5w &= 4 \end{aligned}$$

3x4 linear system

(3 eqn's
4 variables x, y, z, w
linear)

Methods for solving linear systems:

- substitution methods
- elimination methods

Ex:

$$\begin{aligned} 7x-3y&=6 \\ 2x+y&=11 \end{aligned}$$

Substitution:

$$y=\underline{11-2x}$$

$$\downarrow \text{substitute}$$

$$7x-3y=6$$

$$7x-3(11-2x)=6$$

$$7x-33+6x=6$$

$$\frac{13x}{13}=\frac{39}{13}$$

$$\underline{x=3}$$

$$y=11-2x=11-2\cdot 3$$

$$\underline{y=5}$$

Solution:

$$(x,y)=\underline{\underline{(3,5)}}$$

<u>Ex:</u>

$$\begin{cases} 7x - 3y = 6 \\ 2x + y = 11 \quad | \cdot 3 \end{cases}$$

$$\begin{cases} 7x - 3y = 6 \\ 6x + 3y = 33 \end{cases}$$

$$\overline{13x \qquad = 39}$$

$$\begin{cases} 7x - 3y = 6 \\ 13x \qquad = 39 \end{cases}$$

$$\underline{y = 5}$$

$$15 = 3y$$

$$21 - 3y = 6$$

$$7 \cdot 3 - 3y = 6$$

$$\underline{x = 3}$$

<u>Solution:</u> $(x, y) = \underline{\underline{(3, 5)}}$

Geometry of 2×2 linear systems

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

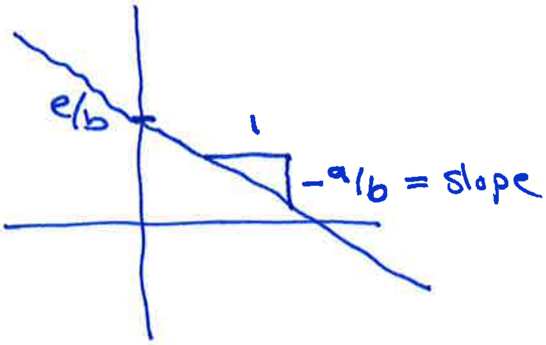
where a, b, c, d, e, f are given numbers

Each eqn. is a straight line

$$ax + by = e$$

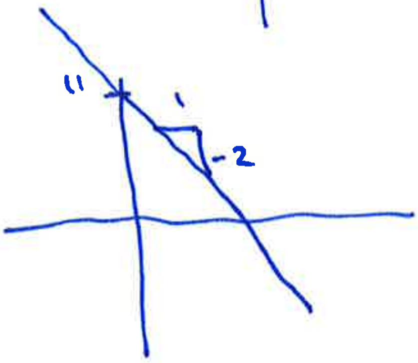
$$\frac{by}{b} = \frac{e - ax}{b}$$

$$\boxed{y = \frac{e}{b} - \frac{a}{b}x}$$



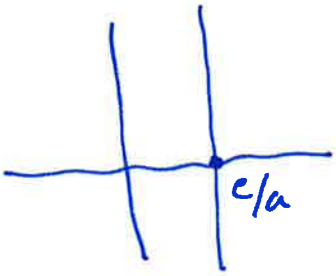
Ex: $2x + y = 11$

$y = 11 - 2x$

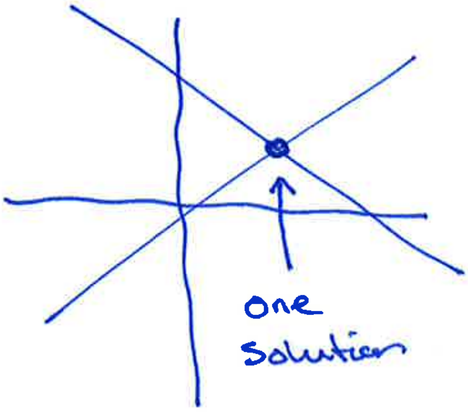


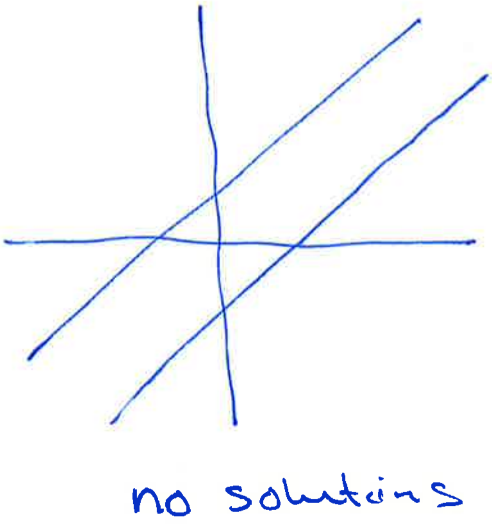
b=0: $ax = e$

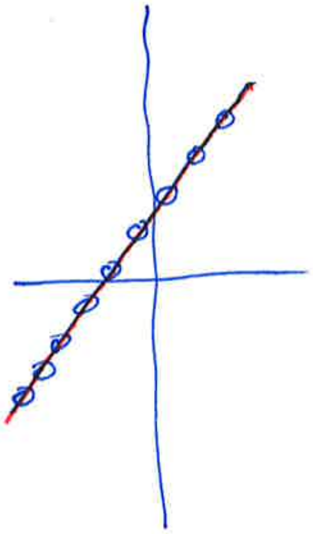
$x = e/a$



$$\left.\begin{aligned} ax+by&=e \\ cx+dy&=f \end{aligned}\right\}$$ general 2×2 linear system







Ex:

$$\begin{aligned} 7x-3y&=6 \\ 2x+y&=11 \end{aligned}$$

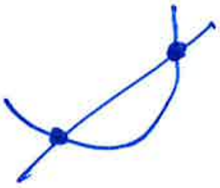
$$\begin{aligned} 4x+2y&=7 \\ 2x+y&=11 \end{aligned}$$

$$\begin{aligned} 4x+2y&=22 \\ 2x+y&=11 \end{aligned}$$

Result:

Any 2×2 linear system has either one solution, infinitely many solutions or no solutions.

Not possible:



Definitions:

A **linear equation** in the variables $x_1, x_2, \dots, x_n$ is an equation that can be written

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

for given numbers $a_1, a_2, \dots, a_n, b$.

An $m \times n$ **linear system** (m equations (linear) in n variables $x_1, \dots, x_n$) is a system of equations that can be written

$$(*) \quad \boxed{\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \;\;\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}}$$

Coefficient matrix:

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n} m$$

Augmented matrix

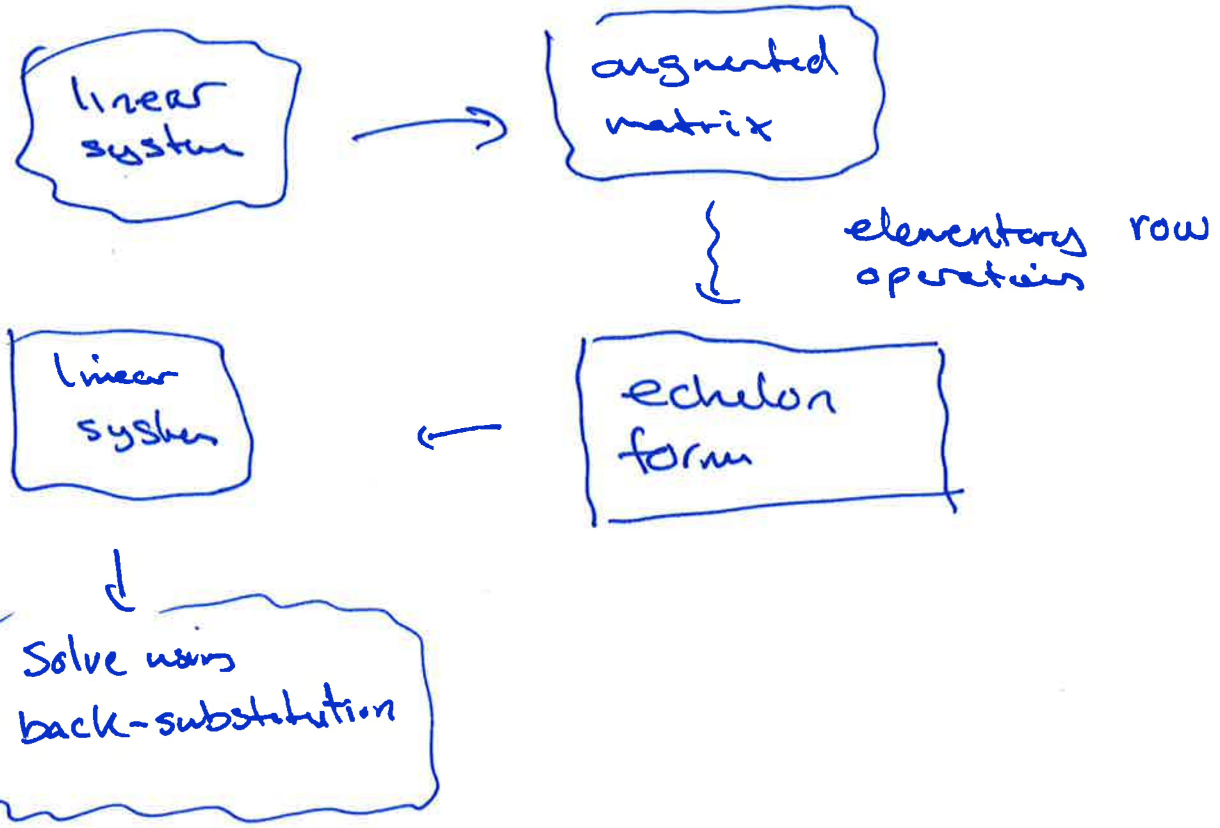
$$(A|\underline{b}) = \left(\begin{array}{c|c} \underbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}}_{n} & \underbrace{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}}_{1} \end{array} \right) m$$

Definition:

A solution of the $m \times n$ linear system $(*)$ is an n-tuple $(x_1, x_2, \dots, x_n) = (s_1, s_2, \dots, s_n)$ that satisfies all m equations simultaneously.

③ Gaussian elimination

- a particular elimination method
- general, efficient, instructive



Ex:

$$\begin{aligned} x+y+z &= 4 \\ x+2y+4z &= 9 \\ x+3y+9z &= 16 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ \textcircled{1} & 2 & 4 & 9 \\ \textcircled{1} & 3 & 9 & 16 \end{array}\right)$$

3x3 linear system

augmented matrix

Echelon form:

leading coefficient or pivot : the first (left most) nonzero number in a row

echelon form: A matrix that satisfies:

i) all zero rows (rows with only 0) must be below all other rows

ii) all entries below a pivot must be zero

Elementary row operations:

i) Interchange (switch) two rows

ii) Multiply a row with a constant $c \neq 0$

iii) Add a multiple of one row to another row

} these operations preserve the solutions of the system

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 16 \end{array}\right) \cdot (-1) \qquad (-1 \;\; -1 \;\; -1 \;|\; -4)$$

need 0's

$$\downarrow \qquad R(2) := R(2) + (-1) \cdot R(1)$$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ 0 & 1 & 3 & 5 \\ 1 & 3 & 9 & 16 \end{array}\right) \cdot (-1) \qquad (-1 \;\; -1 \;\; -1 \;|\; -4)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 2 & 8 & 12 \end{array}\right) \cdot (-2) \qquad (0 \;\; -2 \;\; -6 \;|\; -10)$$

need 0

$$\downarrow$$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 0 & \textcircled{2} & 2 \end{array}\right) \cdot \frac{1}{2}$$

echelon form

Key points:

- start in the upper left corner
- mark pivots
- always use the pivot to set zeros under it

$$\downarrow$$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 4 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 0 & \textcircled{1} & 1 \end{array}\right)$$

echelon form

$$\left(\begin{array}{ccc|c} x & y & z & \\ \textcircled{1} & 1 & 1 & 4 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 0 & \textcircled{2} & 2 \end{array}\right) \longrightarrow \begin{array}{r} \underline{x} + y + z = 4 \\ \underline{y} + 3z = 5 \\ \underline{2z} = 2 \end{array}$$

echelon form

Solution:

$(x,y,z) = \underline{\underline{(1,2,1)}}$

one solution

$$\Longleftarrow \left\{ \begin{array}{l} 2z = 2 \Rightarrow \underline{\underline{z=1}} \\ y + 3z = 5 \Rightarrow y + 3 = 5 \\ \qquad\qquad\qquad \underline{\underline{y=2}} \\ x + y + z = 4 \Rightarrow x + 2 + 1 = 4 \\ \qquad\qquad\qquad \underline{\underline{x=1}} \end{array} \right.$$

Note: The echelon form is not unique.

The pivot positions are unique.

Pivot positions = position of the pivots in an echelon form

Ex:

$$\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ 2x+3y+5z&=5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 5 & 5 \end{array}\right) \begin{array}{l} -1 \quad -2 \\ \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 0 & \textcircled{1} & 3 & 4 \\ 0 & 1 & 3 & -1 \end{array}\right) \begin{array}{l} \\ -1 \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 0 & \textcircled{1} & 3 & 4 \\ 0 & 0 & 0 & \textcircled{-5} \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+z&=3 \\ y+3z&=4 \\ \boxed{0&=-5} \end{aligned}$$

not possible

⇓

<u>no solutions</u>

In general, there are no solutions if and only if there is a pivot position in the last column.

Ex:

$$\begin{aligned} x - y + z + w &= 4 \\ x + y - w &= 5 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \textcircled{1} & -1 & 1 & 1 & 4 \\ 1 & 1 & 0 & -1 & 5 \end{array}\right) \xrightarrow{-1}$$

$$\begin{array}{ccccc} x & y & z & w & \\ \textcircled{1} & -1 & 1 & 1 & 4 \\ 0 & \textcircled{2} & -1 & -2 & 1 \end{array}$$

$$\begin{aligned} \underline{x} - y + z + w &= 4 \\ \underline{2y} - z - 2w &= 1 \end{aligned}$$

echelon form

col's with pivots

x, y: basic variables

z, w: free variables

col's without pivots

Solutions:

$$(x,y,z,w) = \left(-\tfrac{1}{2}z + \tfrac{9}{2},\ \tfrac{1}{2}z + w + \tfrac{1}{2},\ z, w\right)$$

where z, w are free

infinitely many solutions (two degrees of freedom)

$$2y - z - 2w = 1$$

$$\Updownarrow$$

$$\frac{2y}{2} = \frac{z + 2w + 1}{2}$$

$$y = \tfrac{1}{2}z + w + \tfrac{1}{2}$$

$$x - y + z + w = 4$$

$$x = y - z - w + 4$$

$$= \left(\tfrac{1}{2}z + w + \tfrac{1}{2}\right) - z - w + 4$$

$$x = -\tfrac{1}{2}z + \tfrac{9}{2}$$

For the problems for Lecture 1 involving Gauss-Jordan elimination: Either do

i) Use Gaussian elimination instead of Gauss-Jordan elimination, or

ii) Use Gauss-Jordan:

- use elementary row operations until you find a reduced echelon form = echelon form which satisfies two extra conditions:
  - i) all pivots are 1
  - ii) all entries over pivots are 0
- otherwise, Gauss-Jordan uses the same steps as Gaussian elimination.

nonzero coefficient is 1:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3\ x_3 &= 130 \\ x_2 - 0.25x_3 &= 125 \\ x_3 &= 300. \end{aligned} \tag{11}$$

Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11). For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of x_3 in (11b) and obtain $x_2 = 200$. Then, add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$\begin{aligned} x_1 \qquad\qquad &= 300 \\ x_2 \qquad &= 200 \\ x_3 &= 300, \end{aligned} \tag{12}$$

which needs no further work to see the solution. Gauss-Jordan elimination is particularly useful in developing the theory of linear systems; Gaussian elimination is usually more efficient in solving actual linear systems.

Earlier we mentioned a third method for solving linear systems, namely matrix methods. We will study these methods in the next two chapters, when we discuss matrix inversion and Cramer's rule. For now, it suffices to note that all the intuition behind these more advanced methods derives from Gaussian elimination. The understanding of this technique will provide a solid base on which to build your knowledge of linear algebra.

EXERCISES

7.1 Which of the following equations are linear?

a) $3x_1 - 4x_2 + 5x_3 = 6$; *b)* $x_1x_2x_3 = -2$; *c)* $x^2 + 6y = 1$;
d) $(x + y)(x - z) = -7$; *e)* $x + 3^{1/2}z = 4$; *f)* $x + 3z^{1/2} = -4$.

7.2 Solve the following systems by substitution, Gaussian elimination, and Gauss-Jordan elimination:

a)
$$\begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1; \end{aligned}$$

b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 12x_1 + 2x_2 - 3x_3 &= 5 \\ 3x_1 + 4x_2 + x_3 &= -4. \end{aligned}$$

7.3 Solve the following systems by Gauss-Jordan elimination. Note that the third system requires an equation interchange.

$$a)\ \begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned} \qquad b)\ \begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned} \qquad c)\ \begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$$

7.4 Formalize the three elementary equation operations using the abstract notation of system (2), and for each operation, write out the operation which reverses its effect.

7.5 Solve the IS-LM system in Exercise 6.7 by substitution.

7.6 Consider the general IS-LM model with no fiscal policy in Chapter 6. Suppose that $M_s = M^o$; that is, the intercept of the LM-curve is 0.

a) Use substitution to solve this system for Y and r in terms of the other parameters.

b) How does the equilibrium GNP depend on the marginal propensity to save?

c) How does the equilibrium interest rate depend on the marginal propensity to save?

7.7 Use Gaussian elimination to solve

$$\begin{cases} 3x + 3y = 4 \\ -x - y = 10. \end{cases}$$

What happens and why?

7.8 Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

What assumptions do you have to make about the coefficients a_{ij} in order to find a solution?

7.2 ELEMENTARY ROW OPERATIONS

The focus of our concern in the last section was on the coefficients a_{ij} and b_i of the systems with which we worked. In fact, it was a little inefficient to rewrite the x_i's, the plus signs, and the equal signs each time we transformed a system. It makes sense to simplify the representation of linear system (2) by writing two rectangular arrays of its coefficients, called **matrices**. The first array is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

7.11 Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

7.12 Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x - z &= 0. \end{aligned}$$

7.3 SYSTEMS WITH MANY OR NO SOLUTIONS

As we will study in more detail later, the locus of all points (x_1, x_2) which satisfy the linear equation $a_{11}x_1 + a_{12}x_2 = b_1$ is a straight line in the plane. Therefore, the solution (x_1, x_2) of the two linear equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16}$$

is a point which lies on both lines of (16) in the Cartesian plane. Solving system (16) is equivalent to finding where the two lines given by (16) cross. In general, two lines in the plane will be nonparallel and will cross in exactly one point. However, the lines given by (16) can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide, every point on either line is a solution to (16); and (16) has *infinitely* many solutions. An example is the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

In the case where the two parallel lines do not cross, the corresponding system has *no* solution, as the example

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

illustrates. Therefore, it follows from geometric considerations that two linear equations in two unknowns can have one solution, no solution, or infinitely many solutions. We will see later in this chapter that this principle holds for every system of *m linear* equations in *n* unknowns.

$$\left(\begin{array}{ccccccc|c} * & w & w & w & w & w & w & w \\ 0 & 0 & 0 & * & w & w & w & w \\ 0 & 0 & 0 & 0 & * & w & w & w \\ 0 & 0 & 0 & 0 & 0 & 0 & * & w \end{array}\right).$$

This matrix is in row echelon form. The corresponding reduced row echelon form is

$$\left(\begin{array}{ccccccc|c} 1 & w & w & 0 & 0 & w & 0 & w \\ 0 & 0 & 0 & 1 & 0 & w & 0 & w \\ 0 & 0 & 0 & 0 & 1 & w & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & w \end{array}\right).$$

The final solution will have the form

$$\begin{aligned} x_1 &= a_1 - a_2x_2 - a_3x_3 - a_4x_6, \\ x_4 &= b_1 \qquad\qquad\qquad\quad - b_2x_6, \\ x_5 &= c_1 \qquad\qquad\qquad\quad - c_2x_6, \\ x_7 &= d_1. \end{aligned}$$

Here x_7 is the only variable which is unambiguously determined. The variables x_2, x_3, and x_6 are free to take on any values; once values have been selected for these three variables, then values for x_1, x_4, and x_5 are automatically determined.

Some more vocabulary is helpful here. If the jth column of the row echelon matrix $\hat{B}$ contains a pivot, we call x_j a **basic variable.** If the jth column of $\hat{B}$ does not contain a pivot, we call x_j a **free** or **nonbasic variable**. In this terminology, Gauss-Jordan elimination determines a solution of the system in which each basic variable is either unambiguously determined or a linear expression of the free variables. The free variables are free to take on any value. Once one chooses values for the free variables, values for the basic variables are determined.

As in the example above, the free variables are often placed on the right-hand side of the equations to emphasize that their values are not determined by the system; rather, they act as parameters in determining values for the basic variables.

In a given problem which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process and on the order in which the variables are indexed.

EXERCISES

7.13 Reduce the following matrices to row echelon and reduced row echelon forms:

a) $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, *b)* $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}$, *c)* $\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$.

7.14 Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

7.15 Use Gauss-Jordan elimination to determine for what values of the parameter k the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

7.16 Use Gauss-Jordan elimination to solve the following four systems of linear equations. Which variables are free and which are basic in each solution?

$$a)\quad \begin{aligned} w + 2x + y - z &= 1 \\ 3w - x - y + 2z &= 3 \\ - x + y - z &= 1 \\ 2w + 3x + 3y - 3z &= 3; \end{aligned} \qquad b)\quad \begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6 \\ 3w + 2x + 5y - z &= 3; \end{aligned}$$

$$c)\quad \begin{aligned} w + 2x + 3y - z &= 1 \\ -w + x + 2y + 3z &= 2 \\ 3w - x + y + 2z &= 2 \\ 2w + 3x - y + z &= 1; \end{aligned} \qquad d)\quad \begin{aligned} w + x - y + 2z &= 3 \\ 2w + 2x - 2y + 4z &= 6 \\ -3w - 3x + 3y - 6z &= -9 \\ -2w - 2x + 2y - 4z &= -6. \end{aligned}$$

7.17 *a*) Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

b) Suppose you hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

7.18 For what values of the parameter a does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

7.19 From Chapter 6, the stationary distribution in the Markov model of unemployment satisfies the linear system

$$\begin{aligned} (q - 1)x + py &= 0 \\ (1 - q)x - py &= 0 \\ x + y &= 1. \end{aligned}$$