Problems for Lecture 6

BI

1. Find all stationary points and classify them

a) $f(x,y) = e^{xy}$

b) $f(x,y) = e^{x-2y}$

c) $f(x,y) = \sqrt{x^2+y^2+1}$

d) $f(x,y) = x\ln x + y\ln y$

e) (*) $f(x,y) = x\ln(y) - y\ln(x)$ ((*) Difficult.)

2. Solve the Lagrange problems

a) $\max/\min\ f(x,y) = 3x+4y$ when $x^2+y^2=25$

b) $\max\ f(x,y) = y$ when $x^2+y^3=0$

c) $\min\ f(x,y) = 3x^2+4y^2$ when $xy=1$

Solutions for Lecture 6

BI

1. a) $f'_x = ye^{xy}$ $\quad f''_{xx} = y^2 e^{xy}$ $\quad f''_{xy} = (1+xy)e^{xy}$

$f'_y = xe^{xy}$ $\quad f''_{yy} = x^2 e^{xy}$

$f'_x = f'_y = 0$

$y = x = 0 \Rightarrow$ Stat: $(0,0)$

$H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$AC - B^2 = -1 < 0$ saddle pt

b) $u = x - 2y$

$f'_x = e^u \cdot 1$ $\quad f''_{xx} = e^u \cdot 1$ $\quad f''_{xy} = e^u \cdot 1 \cdot (-2)$

$f'_y = e^u \cdot (-2)$ $\quad f''_{yy} = e^u (-2)^2$

$f'_x = f'_y = 0$

$e^{x-2y} = 0 \Rightarrow$ no stat. pts

impossible

c) $u = x^2 + y^2 + 1$

$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{u}}$

$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{y}{\sqrt{u}}$

$f'_x = f'_y = 0$

$x = y = 0 \Rightarrow$ Stat. pts: $(0,0)$

$(u = 1 \neq 0)$

$f''_{xx} = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{2x}{2\sqrt{u}}\right)}{u} \frac{2\sqrt{u}}{2\sqrt{u}} = \frac{2u - 2x^2}{2u\sqrt{u}} = \frac{x^2 + y^2 + 1 - x^2}{u\sqrt{u}} = \frac{y^2+1}{u\sqrt{u}}$

$f''_{xy} = \frac{-x \cdot \frac{1}{2\sqrt{u}} 2x}{u} = \frac{-x^2}{u\sqrt{u}}$

$f''_{yy} = \frac{x^2+1}{u\sqrt{u}}$ $\quad \leftarrow$ symmetry $f(y,x) = f(x,y)$

$f''_{yy}(x,y) = f''_{xx}(y,x)$

$H(f)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (0,0)$ is local min

$AC - B^2 = 1 > 0$, $A = 1 > 0$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

c) $f(x,y) = \sqrt{u}$ with $u = x^2+y^2+1$

$$f'_x = \frac{1}{2\sqrt{u}}\cdot 2x = \frac{2x}{2\sqrt{u}} = \underline{\frac{x}{\sqrt{u}}}$$

$$f'_y = \frac{1}{2\sqrt{u}}\cdot 2y = \frac{2y}{2\sqrt{u}} = \underline{\frac{y}{\sqrt{u}}}$$

$f'_x = f'_y = 0$: $\frac{x}{\sqrt{u}} = 0 \Rightarrow x = 0$

$\frac{y}{\sqrt{u}} = 0 \Rightarrow y = 0$

$(u = \sqrt{1} \neq 0)$

$\Rightarrow$ Stat. pts: $(x,y) = \underline{(0,0)}$

$$f''_{xx} = \left(\frac{x}{\sqrt{u}}\right)'_x = \frac{\left(1\cdot\sqrt{u} - x\cdot\frac{1}{\cancel{2}\sqrt{u}}\cdot\cancel{2}x\right)\cdot\sqrt{u}}{u\cdot\sqrt{u}}$$

$$= \frac{u - x^2}{u\sqrt{u}} = \frac{\cancel{x^2}+y^2+1-\cancel{x^2}}{u\sqrt{u}} = \underline{\frac{y^2+1}{u\sqrt{u}}}$$

$$f''_{xy} = \left(\frac{x}{\sqrt{u}}\right)'_y = \frac{\left(0\cdot\sqrt{u} - x\cdot\frac{1}{\cancel{2}\sqrt{u}}\cdot\cancel{2}y\right)\cdot\sqrt{u}}{u\cdot\sqrt{u}} = \underline{\frac{-xy}{u\sqrt{u}}}$$

$$f''_{yy} = \left(\frac{y}{\sqrt{u}}\right)'_y = \frac{\left(1\cdot\sqrt{u} - y\cdot\frac{1}{\cancel{2}\sqrt{u}}\cdot\cancel{2}y\right)\cdot\sqrt{u}}{u\cdot\sqrt{u}} = \frac{u-y^2}{u\sqrt{u}} = \underline{\frac{x^2+1}{u\sqrt{u}}}$$

$$H(f)(0,0) = \begin{pmatrix} 1/1\sqrt{1} & 0/1\sqrt{1} \\ 0/1\sqrt{1} & 1/1\cdot\sqrt{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det H(f)(0,0) = AC - B^2 = 1 > 0$

$A = 1 > 0$

$\Rightarrow$ $(0,0)$ is a $\underline{\text{local min}}$

BI

d) $f'_x = 1\cdot\ln x + x\cdot\frac{1}{x} = \underline{\ln x + 1}$

$f'_y = \underline{\ln y + 1}$

Stat. pts:

$\ln x + 1 = \ln y + 1 = 0$

$x = y = e^{-1} \Rightarrow \underline{(x,y) = (e^{-1}, e^{-1})}$

$f''_{xx} = 1/x \quad f''_{xy} = 0 \quad f''_{yy} = 1/y$

$H(f)(e^{-1}, e^{-1}) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \Rightarrow (e^{-1}, e^{-1})$ is $\underline{\text{local min}}$

$AC - B^2 = e^2 > 0, \; A = e > 0$

$*$ = Difficult

e) $*$ $f'_x = \ln y - y\cdot\frac{1}{x} = \ln y - \frac{y}{x} = 0$

$f'_y = \cancel{\ } \; x\cdot\frac{1}{y} - \ln x = \frac{x}{y} - \ln x = 0$

Stat. pts:

$\ln y = \frac{y}{x} \Rightarrow x = \frac{y}{\ln y} \Rightarrow \ln\left(\frac{y}{\ln y}\right) = \frac{y/\ln y}{y} = \frac{1}{\ln y}$

$\ln x = \frac{x}{y}$

$\ln y \cdot \ln\left(\frac{y}{\ln y}\right) = 1$

$\ln y \cdot (\ln y - \ln(\ln y)) = 1$

$u(y) = \ln(y)\,(\ln y - \ln(\ln y))$

$u' = \frac{1}{y}(\ln y - \ln(\ln y)) + \ln y \cdot \left(\frac{1}{y} - \frac{1}{\ln y}\cdot\frac{1}{y}\right)$

$= \frac{\ln y - \ln(\ln y) + \ln y - 1}{y}$

$= \frac{2\ln y - \ln(\ln y) - 1}{y}$

To check if $u = 1$ has solution, find out when u is inc. / dec.

$\Updownarrow$

look at sign of u'

$u' = 0$:

$2\ln y - \ln(\ln y) = 1$

$\ln\left(\frac{y^2}{\ln y}\right) = 1$

$\frac{y^2}{\ln y} = e$

$v = \frac{y^2}{\ln y}$

$v' = \frac{2y\ln y - y^2\cdot\frac{1}{y}}{(\ln y)^2}$

$= \frac{y(2\ln y - 1)}{(\ln y)^2}$

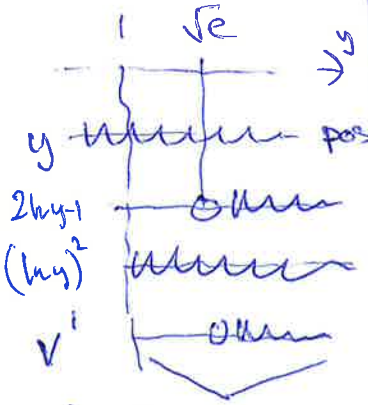
$v' = 0$:

$y\cdot(2\ln y - 1) = 0$

$2\ln y - 1 = 0$

$\ln y = \frac{1}{2}$

$y = e^{1/2} = \sqrt{e}$

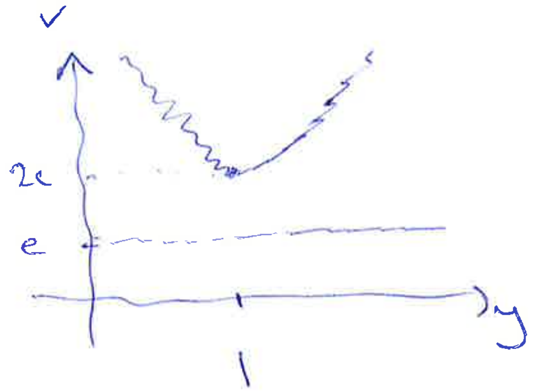


min for v:

$y = \sqrt{e} \Rightarrow v = \frac{e}{1/2} = 2e > e$

To check if $v = e$ has solutions, find out where v is inc. / dec.

$\Rightarrow$ look at sign of v'.



$v(y) = \frac{y^2}{\ln y}$

$v = e$ <u>no solutions</u>

$u' = 0$ <u>no solution</u>

$u' = \frac{2\ln y - \ln(\ln y) - 1}{y}$

$y > 1$: $y > 0$, $2\ln y - \ln(\ln y) - 1$ const. sign since it is never zer

$y = e \rightsquigarrow 2 - \ln 1 - 1$
$= 1 > 0$

$\Downarrow$

$u' > 0$ for all $y > 1$

u increasing fn.

$\Downarrow$

$u = 1$ has <u>at most</u> one solution

u inc. function on $1 < y < \infty$

<u>$y = e$</u> is a solution since

$\ln e\,(\ln(e) - \ln(\ln e)) = 1 \cdot (1 - 0) = 1$

$\Downarrow$

$y = e$ only solution of <u>$u = 1$</u>

$x = \frac{y}{\ln y} = \frac{e}{\ln e} = e$

$\Downarrow$

$(x,y) = (e,e)$ unique stat. pt. of f.

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$

$$H(f)(e,e) = \begin{pmatrix} 1/e & 0 \\ 0 & 1/e \end{pmatrix}$$

$AC - B^2 = 1/e^2 > 0$

$A = 1/e > 0$

$\Downarrow$

$(x,y) = (e,e)$ is <u>local min</u>

2.

a) $\mathcal{L} = 3x + 4y - \lambda \cdot (x^2 + y^2)$

$$\text{FOC}\begin{cases} \mathcal{L}'_x = 3 - \lambda \cdot 2x = 0 & \Rightarrow \quad x = \dfrac{3}{2\lambda} \\ \mathcal{L}'_y = 4 - \lambda \cdot 2y = 0 & \Rightarrow \quad y = \dfrac{4}{2\lambda} \end{cases}$$

$$C \left\{ \quad x^2 + y^2 = 25 \right.$$

$$x^2 + y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 25$$

$$\frac{9+16}{4\lambda^2} = 25$$

$$\frac{25}{4\lambda^2} = 25$$

$$4\lambda^2 = 1$$

$$\lambda^2 = 1/4$$

$$\underline{\lambda = \pm 1/2}$$

$\underline{\lambda = 1/2}$: $x = 3$, $y = 4$

$\Downarrow$

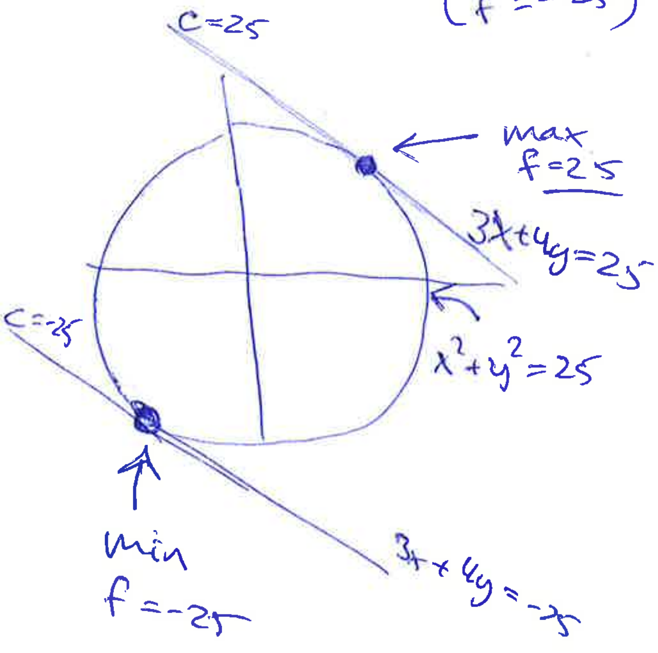
$(x, y; \lambda) = \underline{(3, 4; 1/2)}$

$(f = 25)$

$\underline{\lambda = -1/2}$: $x = -3$, $y = -4$

$(x, y; \lambda) = \underline{(-3, -4; -1/2)}$

$(f = -25)$



inc. values of c
means
lines = level curves move
up and to the right

b) max y when $x^2+y^3=0$

$$h = y - \lambda\cdot(x^2+y^3)$$

FOC $\left\{ \begin{array}{l} h'_x = -\lambda\cdot 2x = 0 \\ h'_y = 1-\lambda\cdot 3y^2 = 0 \end{array} \right.$

C $\{ \quad x^2+y^3=0$

$\Rightarrow \underline{\lambda=0}$ or $\underline{x=0}$

$\lambda = 0$: $1=0$ imp.

$x=0$ $\Downarrow$ $x^2+y^3=0 \Rightarrow \underline{y=0}$

$1-\lambda\cdot 3y^2=0 \Rightarrow 1=0$ imp.

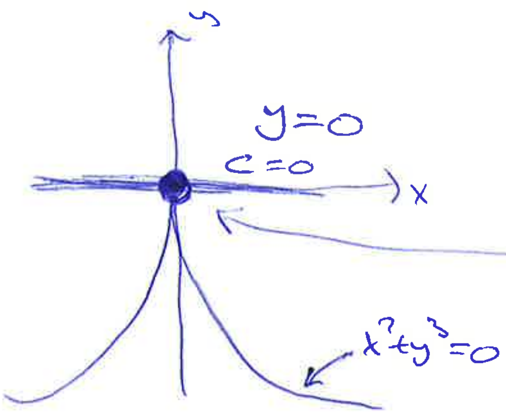
no solution of FOC+C.

$$\boxed{\begin{array}{l} g'_x = 2x = 0 \\ g'_y = 3y^2 = 0 \\ x^2+y^3=0 \end{array}} \qquad \left.\begin{array}{l} x=0 \\ y=0 \\ x=y=0 \text{ ok.} \end{array}\right\}$$

$(0,0)$ is adm. pt. with $g'_x = g'_y = 0$

$\Downarrow$

can be max



inc. values of c means level curve $y=c$ moves up.

Max = 0

c) min $f = 3x^2+4y^2$ when $xy=1$

$L = 3x^2+4y^2 - \lambda \cdot xy$

FOC $\begin{cases} L'_x = 6x - \lambda y = 0 \\ L'_y = 8y - \lambda x = 0 \end{cases}$

C $\{ \; xy = 1$

① $x = \frac{\lambda y}{6}$

② $8y - \lambda \cdot \left(\frac{\lambda y}{6}\right) = 0 \quad | \cdot 6$

$48y - \lambda^2 y = 0$

$y(48 - \lambda^2) = 0$

$\underline{y = 0}$ or $\lambda^2 = 48$

$\underline{\lambda = \pm\sqrt{48}}$

$y=0$	$\lambda = \sqrt{48}$	$\lambda = -\sqrt{48}$
③ $xy = 1$, $x \cdot 0 = 1$ imp. no soln.	① $x = \frac{\sqrt{48}}{6} y$	① $x = -\frac{\sqrt{48}}{6} y$
	③ $xy = \frac{\sqrt{48}}{6} y \cdot y = 1$	③ $xy = -\frac{\sqrt{48}}{6} y \cdot y = 1$
	$y^2 = \frac{6}{\sqrt{48}} = \frac{2 \cdot 3}{\sqrt{4} \cdot \sqrt{12}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{4}} = \sqrt{3/4}$	$y^2 = -\frac{6}{\sqrt{48}}$ imp. no soln.
	$\underline{y = \pm\sqrt[4]{3/4}}$	
	$x = \pm\sqrt[4]{3/4} \cdot \frac{\sqrt{48}}{6} = \pm\sqrt[4]{4/3}$	

$\pm\sqrt[4]{3/4} \cdot \frac{\sqrt{48}}{6}$

$= \pm\sqrt[4]{\frac{3}{4}} \cdot \sqrt[4]{\left(\frac{4}{3}\right)^2}$

$= \pm\sqrt[4]{4/3}$

Pts:

$\left(\sqrt[4]{4/3}, \sqrt[4]{3/4}; \sqrt{48}\right)$

$\left(-\sqrt[4]{4/3}, -\sqrt[4]{3/4}; \sqrt{48}\right)$

$f = 3 \cdot \sqrt{4/3} + 4 \cdot \sqrt{3/4}$

$\simeq 6.93$ min pt./value

inc. values of c means the level curve = ellipse $3x^2+4y^2 = c$ moves outwards ("radius" increases)