Problems for Lecture 5

1. Compute the integrals

a) $\int x^2 \ln x \, dx$

b) $\int \frac{x^3 + 2x^2 + 1}{x} dx$

c) $\int x e^{-x^2} dx$

d) $\int \frac{x}{x^2+1} dx$

e) $\int \frac{1}{(2x-3)^2} dx$

f) $\int x^3 e^{-x^2} dx$

g) $\int e^{\sqrt{x}} dx$

2. Simplify the expressions using polynomial division:

a) $\frac{x^3}{x^2-1}$

b) $\frac{x^2+2x-3}{x+1}$

c) $\frac{x^3+x^2+x+1}{x+1}$

d) $\frac{x^4+1}{x-1}$

3. Compute the integrals:

a) $\int \frac{3}{2x-4} dx$

b) $\int \frac{1}{x^2+x} dx$

c) $\int \frac{x}{x^2-4x-5} dx$

d) $\int \frac{x^2}{x^2-1} dx$

e) $\int \frac{1}{x^2-4x+4} dx$

4. Find the first order partial derivatives and the Hessian:

a) $f(x,y) = x^3 - 3xy + y^3$

b) $f(x,y) = e^{xy}$

c) $f(x,y) = e^{x+2y}$

d) $f(x,y) = \sqrt{x^2+y^2+1}$

e) $f(x,y) = \ln(x^2+y^2+4)$

f) $f(x,y) = \ln(xy) - 1$

g) $f(x,y) = x\ln y - y\ln x$

Solutions for Lecture 5.

1. a) $\int \underset{\overset{\shortparallel}{v'}}{x^2} \underset{\overset{\shortparallel}{u}}{\ln x}\, dx = \underset{\overset{\shortparallel}{v}}{\frac{1}{3}x^3} \cdot \underset{\overset{\shortparallel}{u}}{\ln x} - \int \underset{\overset{\shortparallel}{v}}{\frac{1}{3}x^3} \cdot \underset{\overset{\shortparallel}{u'}}{\frac{1}{x}}\, dx$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \underline{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

b) $\int \frac{x^3+2x^2+1}{x} dx = \int x^2 + 2x + \frac{1}{x}\, dx = \underline{\frac{1}{3}x^3 + x^2 + \ln|x| + C}$

c) $\int x e^{-x^2} dx = \int \cancel{x} e^u \cdot \frac{du}{-2\cancel{x}} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C$

$\boxed{\begin{array}{c} u = -x^2 \\ du = -2x\,dx \end{array}}$

$$= \underline{-\frac{1}{2} e^{-x^2} + C}$$

d) $\int \frac{x}{x^2+1} dx = \int \frac{x}{u} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C$

$\boxed{\begin{array}{c} u = x^2+1 \\ du = 2x\,dx \end{array}}$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$= \underline{\frac{1}{2}\ln(x^2+1) + C}$$

e) $\int \frac{1}{(2x-3)^2} dx = \int \frac{1}{u^2}\frac{du}{2} = \frac{1}{2}\int u^{-2} du$

$\boxed{\begin{array}{c} u = 2x-3 \\ du = 2\,dx \end{array}}$

$$= \frac{1}{2}\left(-\frac{1}{u}\right) + C$$

$$= \underline{-\frac{1}{2} \cdot \frac{1}{2x-3} + C}$$

f) $\int x^3 e^{-x^2} dx = \int x^3 e^u \frac{du}{-2x} = \int -\frac{1}{2}x^2 e^u du$

$\boxed{\begin{array}{c} u = -x^2 \\ du = -2x\,dx \end{array}}$

$$= -\frac{1}{2}\int (-u) e^u du$$

$x^2 = -u$

$$= \frac{1}{2}\int u e^u du \underset{\substack{\uparrow \\ \text{int. by} \\ \text{parts}}}{=} u e^u - \int 1 \cdot e^u du = u e^u - e^u + C$$

$$= \underline{-x^2 e^{-x^2} - e^{-x^2} + C}$$

g) $\int e^{\sqrt{x}}\,dx = \int e^{u} \cdot 2\sqrt{x}\,du$

$\boxed{\begin{array}{l} u=\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$ $= \int e^{u} \cdot 2u\,du$

$\sqrt{x}=u$

$= \int 2ue^{u}\,du \underset{\text{int. by parts}}{=} 2ue^{u} - \int 2e^{u}\,du$

$= 2ue^{u} - 2e^{u} + C = \underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

2\. a) $x^{3} : x^{2}-1 = x \Rightarrow \underline{\frac{x^{3}}{x^{2}-1} = x + \frac{x}{x^{2}-1}}$

$-(x^{3}-x)$

x

b) $x^{2}+2x-3 : x+1 = x+1 \Rightarrow \frac{x^{2}+2x-3}{x+1} = \underline{x+1+\frac{-4}{x+1}}$

$-(x^{2}+x\ )$

$x-3$

$-(x+1)$

-4

c) $x^{3}+x^{2}+x+1 : x+1 = x^{2}+1 \Rightarrow \frac{x^{3}+x^{2}+x+1}{x+1} = \underline{x^{2}+1}$

$-(x^{3}+x^{2}\ )$

$x+1$

$x+1$

0

d) $x^{4}+1 : x-1 = x^{3}+x^{2}+x+1 \Rightarrow \frac{x^{4}+1}{x-1} = \underline{x^{3}+x^{2}+x+1+\frac{2}{x-1}}$

$-(x^{4}-x^{3})$

$x^{3}+1$

$-(x^{3}-x^{2})$

$x^{2}+1$

$-(x^{2}-x)$

$x+1$

$-(x-1)$

2

3. a) $\int \frac{3}{2x-4}\,dx = \int \frac{3}{u}\frac{du}{2} = \frac{3}{2}\int \frac{1}{u}\,du$

$u = 2x-4$
$du = 2dx$

$= \underline{\frac{3}{2}\ln|2x-4| + C}$

b) $\int \frac{1}{x^2+x}\,dx \underset{\uparrow}{=} \int \frac{1}{x} + \frac{-1}{x+1}\,dx = \underline{\ln|x| - \ln|x+1|}$ $\underline{+C}$

$\frac{1}{x^2+x} = \frac{1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big| \cdot (x+1)x$

$1 = A\cdot(x+1) + Bx$

$= (\underset{\overset{\shortparallel}{0}}{A+B})x + (\underset{\overset{\shortparallel}{1}}{A})$

$\underline{A=1}, \quad \underline{B=-1}$

$= \ln\left|\frac{x}{x+1}\right| + C$

$\uparrow$

rules for logarithms

c) $\int \frac{x}{x^2-4x-5}\,dx \underset{\uparrow}{=} \int \frac{5/6}{x-5} + \frac{1/6}{x+1}\,dx$

$x^2-4x-5 = (x-5)(x+1)$

$\frac{x}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1} \quad \Big| \cdot (x-5)(x+1)$

$x = A(x+1) + B(x-5)$

$= (\underset{\overset{\shortparallel}{1}}{A+B})x + (\underset{\overset{\shortparallel}{0}}{A-5B})$

$A+B=1 \qquad \nwarrow 5B+B=1$
$A-5B=0 \Rightarrow A=5B$

$6B=1$

$A=\underline{5/6} \qquad B=\underline{1/6}$

$= \underline{\frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C}$

$= \frac{1}{6}\left(\ln|x-5|^5 + \ln|x+1|\right) + C$

$= \underline{\frac{1}{6}\ln\left|(x-5)^5(x+1)\right| + C}$

d) $\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = x + \int \frac{1}{(x-1)(x+1)}\,dx$

$x^2 : x^2-1 = 1$

$-\,(x^2-1)$

$\overline{\quad 1 \quad}$

$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Big| \cdot (x-1)(x+1)$

$1 = A(x+1) + B(x-1)$

$= \underset{\substack{\shortparallel \\ 0}}{(A+B)}x + \underset{\substack{\shortparallel \\ 1}}{(A-B)}$

$A + B = 0$

$A - B = 1$

$\overline{2A \quad = 1} \longrightarrow \underline{A = 1/2}, \; B = \underline{-1/2}$

$= x + \int \frac{1/2}{x-1} + \frac{-1/2}{x+1}\,dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= \underline{x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$

e) $\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$

$u = x-2$

$du = 1 \cdot dx$

$= \underline{-\frac{1}{x-2} + C}$

4.

a) $f = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$f''_{xx} = 6x$

$f''_{xy} = -3$

$f''_{yy} = 6y$

$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

↑ know that $f''_{yx} = f''_{xy} = -3$

b) $f = e^{xy}$

$f'_x = e^{xy} \cdot (xy)'_x = ye^{xy}$

$f'_y = e^{xy} \cdot (xy)'_y = xe^{xy}$

$f''_{xx} = y \cdot ye^{xy} = y^2 e^{xy}$

$f''_{xy} = 1 \cdot e^{xy} + y \cdot e^{xy} \cdot x$
$= (xy+1)e^{xy}$

$f''_{yy} = x \cdot x e^{xy} = x^2 e^{xy}$

$$H(f) = \begin{pmatrix} y^2 e^{xy} & (xy+1)e^{xy} \\ (xy+1)e^{xy} & x^2 e^{xy} \end{pmatrix}$$

$$= \begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix} \cdot e^{xy}$$

c) $f = e^{x+2y}$

$f'_x = e^{x+2y}$

$f'_y = 2e^{x+2y}$

$f''_{xx} = e^{x+2y}$

$f''_{xy} = 2e^{x+2y}$

$f''_{yy} = 4e^{x+2y}$

$H(f) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} e^{x+2y}$

d) $f = \sqrt{x^2+y^2+1}$

$$f'_x = \frac{1}{2\sqrt{x^2+y^2+1}} \cdot 2x = \frac{2x}{2\sqrt{--}} = \frac{x}{\sqrt{x^2+y^2+1}}$$

$$f'_y = \frac{1}{2\sqrt{\sim}} \cdot 2y = \frac{2y}{2\sqrt{--}} = \frac{y}{\sqrt{x^2+y^2+1}}$$

$u = x^2+y^2+1$: $\quad f'_x = \frac{x}{\sqrt{u}} \quad f'_y = \frac{y}{\sqrt{u}}$

$$f''_{xx} = \frac{1\cdot\sqrt{u} - x\cdot\frac{1}{2\sqrt{u}}\cdot\cancel{2}x}{u} = \frac{\sqrt{u}-\frac{x^2}{\sqrt{u}}}{u}\cdot\frac{\sqrt{u}}{\sqrt{u}} = \frac{u-x^2}{u\sqrt{u}} = \frac{\cancel{x^2}+y^2+1-\cancel{x^2}}{(x^2+y^2+1)^{3/2}}$$

$$= \underline{\frac{y^2+1}{(x^2+y^2+1)^{3/2}}}$$

$$f''_{xy} = \frac{0\cdot\sqrt{u} - x\cdot\frac{1}{\cancel{2}\sqrt{u}}\cdot\cancel{2}y}{u} = \frac{-xy}{u\sqrt{u}} = \frac{-xy}{(x^2+y^2+1)^{3/2}}$$

$$f''_{yy} = \underline{\frac{x^2+1}{(x^2+y^2+1)^{3/2}}} \longleftarrow$$

By symmetry: $f(x,y) = f(y,x)$

so $f''_{yy}(x,y) = f''_{xx}(y,x)$

(i.e. we swap x and y)

$$H(f) = \frac{1}{(x^2+y^2+1)^{3/2}}\cdot\begin{pmatrix} y^2+1 & -xy \\ -xy & x^2+1 \end{pmatrix}$$

e) $f = \ln(x^2+y^2+4) = \ln(u)$, $u = x^2+y^2+4$

$$f'_x = \frac{2x}{x^2+y^2+4} \qquad f''_{xx} = \frac{2(x^2+y^2+4)-2x(2x)}{u^2} = \frac{-2x^2+2y^2+4}{(x^2+y^2+4)^2}$$

$$f'_y = \frac{2y}{x^2+y^2+4} \qquad f''_{xy} = \frac{0\cdot\cancel{x} - 2x\cdot 2y}{u^2} = \frac{-4xy}{(x^2+y^2+4)^2}$$

$$f''_{yy} = \frac{2x^2-2y^2+4}{(x^2+y^2+4)^2} \longleftarrow \text{by symmetry}$$

$$H(f) = \frac{1}{(x^2+y^2+4)}\cdot\begin{pmatrix} -2x^2+2y^2+4 & -4xy \\ -4xy & 2x^2-2y^2+4 \end{pmatrix}$$

$f = \ln(xy) - 1 = \ln x + \ln y - 1$

f) $f'_x = \frac{1}{x}$ $\quad f''_{xx} = -1/x^2$ $\quad f''_{xy} = 0$ $\quad H(f) = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$

$f'_y = \frac{1}{y}$ $\quad f''_{yy} = -1/y^2$

g) $f = x\ln y - y\ln x$

$f'_x = \ln y - \frac{y}{x}$ $\quad f''_{xx} = -y \cdot (-1/x^2) = \frac{y}{x^2}$

$f'_y = \frac{x}{y} - \ln x$ $\quad f''_{xy} = \frac{1}{y} - \frac{1}{x}$

$f''_{yy} = x \cdot (-1/y^2) = -\frac{x}{y^2}$

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$