Plan:

1. Partial derivatives and the Hessian matrix
2. Unconstrained max/min-problems
3. Lagrange problems

Reading: [ME] Ch. 17–18

Review: Integration

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)}$$

Ex:

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C = \underline{\frac{2}{3}x^{3/2} + C}$$

$$\int \frac{1}{x}\,dx = \underline{\ln|x| + C}$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = \underline{\underline{-\frac{1}{x} + C}}$$

$$\left(\begin{aligned} &\ln x^2 + C \neq \\ &(\ln x^2)' = \frac{1}{x^2}\cdot 2x \\ &\qquad = 2/x \end{aligned}\right)$$

(a) Substitution:

Ex: $\int x e^{-x^2} dx = \int \cancel{x} \cdot e^u \cdot \frac{du}{-2\cancel{x}}$

$u = -x^2$
$du = -2x\, dx$

$= -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C$

(b) Integration by parts: $\int u'v\,dx = uv - \int uv'\,dx$

Ex: $\int x^2 \cdot \ln x\, dx = \frac{1}{3}x^3 \cdot \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx$

$\cancel{\frac{1}{3}x^3 \cdot \int \ln x\, dx}$ $\neq$

$$\boxed{\begin{array}{ll} u = x^3/3 & v = \ln x \\ u' = x^2 & v' = 1/x \end{array}}$$

$\neq$ $\cancel{\frac{1}{3} \cdot \frac{x^4}{4} \ln x}$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \frac{1}{3}x^3 + C$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(c) Integration of rational expressions:

Ex: $\int \frac{x^2+1}{x} dx = \int x + \frac{1}{x}\, dx = \frac{1}{2}x^2 + \ln|x| + C$

$\int \frac{2x}{x^2-1} dx = ?$

$$\frac{2x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$ ← find A, B (constants)

factorize the denominator:

$$\underline{x^2-1 = (x-1)\cdot(x+1)}$$

$$\frac{2x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Big| \cdot (x^2-1)$$

$$2x = \frac{A\cdot\cancel{(x^2-1)}^{x+1}}{\cancel{x-1}} + \frac{B\,\cancel{(x^2-1)}^{x-1}}{\cancel{x+1}}$$

$$2x = A\cdot(x+1) + B(x-1)$$
$$= Ax + Bx + A - B$$
$$2x = (A+B)x + (A-B)$$

$$A+B = 2 \qquad 2B = 2$$
$$A-B = 0 \qquad A = B$$

$$\underline{B=1},\ \underline{A=1}$$

Conclusion:

$$\frac{2x}{x^2-1} = \frac{1}{x-1} + \frac{1}{x+1}$$

$$\int \frac{2x}{x^2-1}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx$$
$$= \underline{\ln|x-1| + \ln|x+1| + C}$$

Note: Rational expressions

(a) If deg num. $\geq$ deg denom. then do polynomial div.

(b) $$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\cdot \ln|ax+b| + C$$

if degree denom $= 1$

(c) can use substitution

$u = x^2-1$

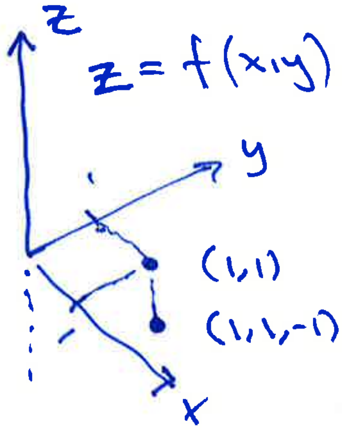
$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\cdot\frac{du}{a} = \int \frac{A}{a}\cdot\frac{1}{u}\,du = \frac{A}{a}\ln|u| + C$$
$$= \frac{A}{a}\ln|ax+b| + C$$

$$\boxed{\begin{aligned} u &= ax+b \\ du &= a\cdot dx \end{aligned}}$$

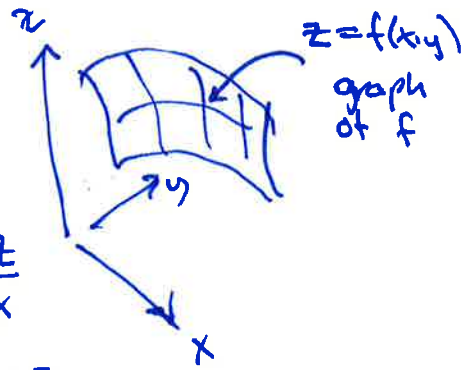
① Functions in two variables

$$f(x,y) = x^3 - 3xy + y^3$$

$$f(1,1) = 1^3 - 3\cdot 1 \cdot 1 + 1^3 = -1$$



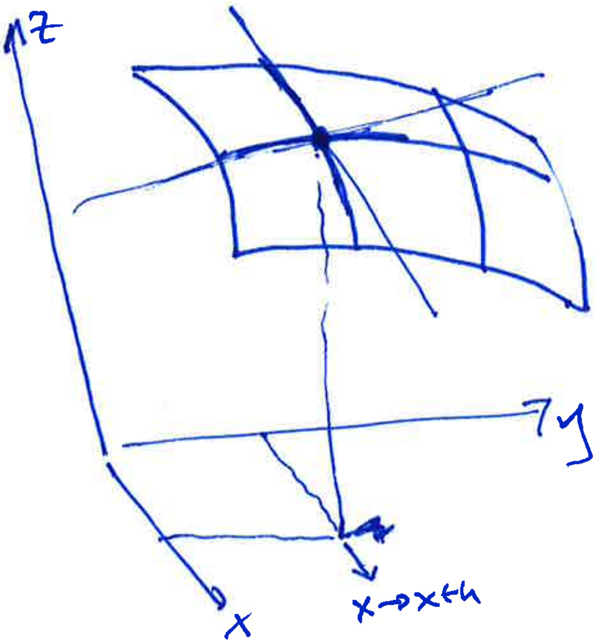
Derivation of fn. in two variables = partial derivation



$$\boxed{\begin{aligned} f'_x &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ f'_y &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned}}$$

$\leftarrow \frac{\Delta z}{\Delta x}$

$\leftarrow \frac{\Delta z}{\Delta y}$



$f(x+1,y) \simeq f(x,y) + f'_x \cdot 1$

$f(x,y+1) \simeq f(x,y) + f'_y \cdot 1$

Notation:

$$f'_x = \frac{\partial f}{\partial x} = (x^3-3xy+y^3)'_x \quad \cancel{\neq f'(x)}$$

$f = x^3 - 3xy + y^3$

$$f'_y = \frac{\partial f}{\partial y} = (x^3-3xy+y^3)'_y$$

How to compute partial derivatives:

Ex: $f = x^3 - 3xy + y^3$

y is const. $\longrightarrow$

$$f'_x = (x^3)'_x - 3\cdot(xy)'_x + (y^3)'_x = 3x^2 - 3(xy)'_x$$

$$+ 0 = 3x^2 - 3(1\cdot y + x\cdot 0) = \underline{3x^2 - 3y}$$

x is const. $\longrightarrow$

$$f'_y = (x^3-3xy+y^3)'_y = 0 - 3x\cdot 1 + 3y^2$$

$$= \underline{-3x + 3y^2}$$

Ex: $f(x,y) = e^{x-y} = e^u, \quad u = x-y$

$$f'_x = e^u\cdot(x-y)'_x = e^u\cdot 1 = \underline{e^{x-y}}$$

$$f'_y = e^u\cdot(x-y)'_y = e^u\cdot(-1) = \underline{-e^{x-y}}$$

Second order partial derivatives: and the Hessian matrix

Ex: $f = x^3 - 3xy + y^3$

$f'_x(x,y) = f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$f''_{xx} = (3x^2 - 3y)'_x = 6x$

$f''_{xy} = (3x^2 - 3y)'_y = -3$

$f''_{yx} = (-3x + 3y^2)'_x = -3$

$f''_{yy} = (-3x + 3y^2)'_y = 6y$

$$H(f) = H(f)(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Hessian matrix

Result: If f is "nice" (C^2), then $f''_{xy} = f''_{yx}$
$\Updownarrow$
$H(f)$ is symmetric.

② Unconstrained optimization:

$$\boxed{\max/\min \quad f(x,y)}$$

Ex: $\min f(x,y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f'_x &= 3x^2 - 3y = 0 \\ f'_y &= -3x + 3y^2 = 0 \end{aligned}\right\}$$ First order conditions (FOC)

Stationary pts = all pts such that $f'_x = f'_y = 0$

$3x^2 - 3y = 0 \longrightarrow \frac{3y}{3} = \frac{3x^2}{3}$ $\quad y = x^2$

$-3x + 3y^2 = 0$

$-3x + 3y^2 = 0$

$-3x + 3(x^2)^2 = 0$

$-3x + 3x^4 = 0$

$3x(-1 + x^3) = 0$

$3x = 0$ or $-1 + x^3 = 0$

$x = 0$ $\quad x^3 = 1$

$y = 0^2 = 0$ $\quad x = \sqrt[3]{1} = 1$

$x = 1$

$y = 1^2 = 1$

Stationary pts:

$(x,y) = (0,0), (1,1)$

Result: If f is "nice", then we have:

(x,y) is max/min for f $\Rightarrow$ (x,y) is a stationary pt.

Conclusion: Find stationary pts of f

$\rightsquigarrow$ candidates for max/min

Ex: $f = x^3 - 3xy + y^3$ Stationary pts: $(0,0)$, $(1,1)$

= candidates for min

Local classification:

stationary pt (x^*, y^*) → local max / local min / saddle pt = not local max, not local min.

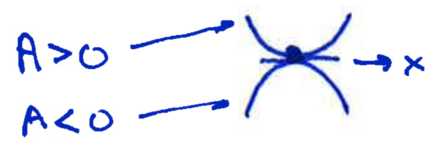
$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

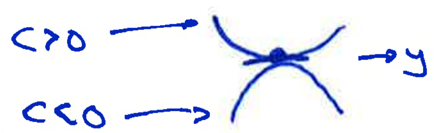
$f''_{yx}(x^*,y^*)$

det

$A = f''_{xx}(x^*,y^*)$ $B = f''_{xy}(x^*,y^*)$

$C = f''_{yy}(x^*,y^*)$





Second derivative test: stationary pt. (x^*,y^*)

$AC - B^2 > 0,\ A > 0$: (x^*,y^*) is local min

$AC - B^2 > 0,\ A < 0$: (x^*,y^*) is local max

$AC - B^2 < 0$: (x^*,y^*) is saddle pt

$AC - B^2 = 0$: no conclusion

$\boxed{\min f(x,y)}$

Ex: $f = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

Stationary pts: $(0,0)$, $(1,1)$

Second derivative test:

$(x,y) = (0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$A = 0$
$B = -3$
$C = 0$

$\underbrace{\det H(f)(0,0)}_{AC - B^2} = 0 - 9 = -9 < 0$

$\Downarrow$

$(0,0)$ is saddle pt, not min

$(x,y) = (1,1)$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

$A = 6$
$B = -3$
$C = 6$

$\underbrace{\det H(f)(1,1)}_{AC - B^2} = 6^2 - (-3)^2$

$= 36 - 9 = 27 > 0$

$\underset{6}{\overset{}{A}} > 0 \Rightarrow$ (1,1) local min

Concl: One cand. for min, $(x,y) = (1,1)$ with $f = -1$
local min

$f(-2,0) = -8 \Rightarrow (1,1)$ is not global min

$\Downarrow$

there is no global min

Summary: Method for max/min $f(x,y)$:

1. Find stationary pts: $f'_x = f'_y = 0$
2. For each —"—, classify it as local max, local min, saddle pt.
3. Try to figure out if any of the local max/min are also global max/min.

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3 Lagrange problems: max/min $f(x,y)$ when $g(x,y)=a$

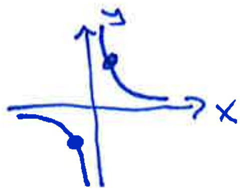
$f(x,y)$ ↑ objective fn.

$g(x,y)=a$ ↑ equality constraint

Ex: min x^2+y^2 when $\underline{xy=1}$ ↗ $y = 1/x$

$x^2+y^2 = f(x,y)$ $\qquad g(x,y) = xy$, $a = 1$

$(xy-1=0)$ $\quad g(x,y) = xy-1$, $a=0$



Candidate pts: $\mathcal{L} = f(x,y) - \lambda \cdot g(x,y)$
$= x^2+y^2 - \lambda \cdot (xy)$

FOC: $\begin{cases} \mathcal{L}'_x = 2x - \lambda \cdot y = 0 \\ \mathcal{L}'_y = 2y - \lambda \cdot x = 0 \end{cases}$ ⇝ $\boxed{\text{Solution = candidate pts.}}$

C: $\{ \quad xy = 1$

$2x = \lambda y$
$\underline{\underline{x = \frac{\lambda y}{2}}}$

$2y = \lambda x = \lambda \cdot \frac{\lambda y}{2}$
$2y = \frac{\lambda^2}{2} y \quad | \cdot 2$

$4y = \lambda^2 y$
$4y - \lambda^2 y = 0$
$(4-\lambda^2)y = 0$

$\cancel{y=0}$ or $\lambda = \pm 2$

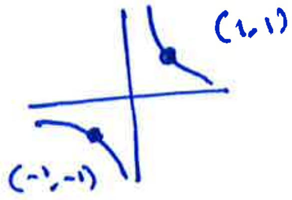
$\underline{\lambda = 2}$: $x = y = \pm 1$ $\quad \boxed{(1,1),(-1,-1)}$

$\lambda = -2$: $x = -y$ $\quad \cancel{-y^2 = 1}$

Candidate pts = Solutions of FOC + C :

$(x,y;\lambda) = (1,1;2)$ $f=2$

$(-1,-1;2)$ $f=2$



Result: When $f(x,y)$, $g(x,y)$ are "nice", then:

(x^*,y^*) is max/min in Lagrange problem $\Rightarrow$ $(x^*,y^*;\lambda^*)$ is a solution of FOC + C for some λ^*.

Must determine if any of the candidate pts are max/min in the Lagrange problem.

The first order leading principal minor is $F_{xx} = 6x$ and the second order leading principal minor is $\det D^2F(\mathbf{x}) = -36xy - 81$. At $(0, 0)$, these two minors are 0 and -81, respectively. Since the second order leading principal minor is negative, $(0, 0)$ is a saddle of F — neither a max nor a min. At $(3, -3)$, these two minors are 18 and 243. Since these two numbers are positive, $D^2F(3, -3)$ is positive definite and $(3, -3)$ is a strict local min of F.

Notice that $(3, -3)$ is not a *global* min, because at the point $(0, n)$, $F(0, n) = -n^3$, which goes to $-\infty$ as $n \to \infty$.

EXERCISES stationary pts.

17.1 For each of the following functions defined on $\mathbf{R}^2$, find the critical points and classify these as local max, local min, saddle point, or "can't tell":

a) $x^4 + x^2 - 6xy + 3y^2$, *b)* $x^2 - 6xy + 2y^2 + 10x + 2y - 5$,
c) $xy^2 + x^3y - xy$, *d)* $3x^4 + 3x^2y - y^3$.

17.2 For each of the following functions defined on $\mathbf{R}^3$, find the critical points and classify them as local max, local min, saddle point, or "can't tell":
a) $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$,
b) $(x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$.

17.4 GLOBAL MAXIMA AND MINIMA

The first and second order sufficient conditions of the last section will find all the local maxima and minima of a differentiable function whose domain is an open set in $\mathbf{R}^n$. As Example 17.2 illustrates, these conditions say nothing about whether or not any of these local extrema is a *global* max or min. In this section, we will discuss sufficient conditions for global maxima and minima of a real-valued function on $\mathbf{R}^n$.

The study of one-dimensional optimization problems in Section 3.5 put forth two conditions for a critical point x^* of f to be a global max (or min), when f is a C^2 function defined on a connected interval I of $\mathbf{R}^1$:

(1) x^* is a local max (or min) and it's the only critical point of f in I; or
(2) $f'' \leq 0$ on all of I (or $f'' \geq 0$ on I for a min), that is, f is a concave function on I (or f is a convex function for a min).

Condition 1 does not work in higher dimensions, as the function F whose level sets are pictured in Figure 17.1 illustrates. The point A in Figure 17.1 is a local max of F in the open set U. Even though A is the only critical point of F in U, the function F takes on a higher value at point B.

Problems for Lecture 6

BI

1. Find all stationary points and classify them

a) $f(x,y) = e^{xy}$

b) $f(x,y) = e^{x-2y}$

c) $f(x,y) = \sqrt{x^2+y^2+1}$

d) $f(x,y) = x\ln x + y\ln y$

e) ✱ $f(x,y) = x\ln(y) - y\ln(x)$ (✱ Difficult.)

2. Solve the Lagrange problems

a) $\max/\min\ f(x,y) = 3x+4y$ when $x^2+y^2=25$

b) $\max\ f(x,y) = y$ when $x^2+y^3=0$

c) $\min\ f(x,y) = 3x^2+4y^2$ when $xy=1$

Solutions for Lecture 6

1. a) $f'_x = y e^{xy}$ $\quad$ $f''_{xx} = y^2 e^{xy}$ $\quad$ $f''_{xy} = (1+xy) e^{xy}$

$f'_y = x e^{xy}$ $\quad$ $f''_{yy} = x^2 e^{xy}$

$f'_x = f'_y = 0$ $\quad$ $H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$y = x = 0 \Rightarrow$ Stat: $(0,0)$ $\quad$ $AC - B^2 = -1 < 0$ <u>Saddle pt</u>

b) $u = x - 2y$

$f'_x = e^u \cdot 1$ $\quad$ $f''_{xx} = e^u \cdot 1$ $\quad$ $f''_{xy} = e^u \cdot 1 \cdot (-2)$

$f'_y = e^u \cdot (-2)$ $\quad$ $f''_{yy} = e^u (-2)^2$

$f'_x = f'_y = 0$

$e^{x-2y} = 0 \Rightarrow$ <u>no stat. pts</u>

impossible

c) $u = x^2 + y^2 + 1$

$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{u}}$

$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{y}{\sqrt{u}}$

$f'_x = f'_y = 0$

$x = y = 0 \Rightarrow$ Stat. pts: $(0,0)$

$(u = 1 \neq 0)$

$$f''_{xx} = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{2x}{2\sqrt{u}}\right) 2\sqrt{u}}{u \quad 2\sqrt{u}} = \frac{2u - 2x^2}{2u\sqrt{u}} = \frac{x^2+y^2+1-x^2}{u\sqrt{u}} = \frac{y^2+1}{u\sqrt{u}}$$

$$f''_{xy} = \frac{-x \cdot \frac{1}{2\sqrt{u}} \cdot 2y}{u} = \frac{-xy}{u\sqrt{u}}$$

$$f''_{yy} = \frac{x^2+1}{u\sqrt{u}}$$

$\leftarrow$ symmetry $f(y,x) = f(x,y)$

$f''_{yy}(x,y) = f''_{xx}(y,x)$

$H(f)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (0,0)$ is <u>local min</u>

$AC - B^2 = 1 > 0$, $A = 1 > 0$

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c) $f(x,y) = \sqrt{u}$ with $u = x^2+y^2+1$

$$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{u}} = \underline{\frac{x}{\sqrt{u}}}$$

$$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{2y}{2\sqrt{u}} = \underline{\frac{y}{\sqrt{u}}}$$

$f'_x = f'_y = 0$: $\frac{x}{\sqrt{u}} = 0 \Rightarrow x = 0$

$\frac{y}{\sqrt{u}} = 0 \Rightarrow y = 0$

$(u = \sqrt{1} \neq 0)$

$\Rightarrow$ Stat. pts: $(x,y) = \underline{(0,0)}$

$$f''_{xx} = \left(\frac{x}{\sqrt{u}}\right)'_x = \frac{\left(1\cdot\sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}x\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}}$$

$$= \frac{u - x^2}{u\sqrt{u}} = \frac{\cancel{x^2} + y^2 + 1 - \cancel{x^2}}{u\sqrt{u}} = \underline{\frac{y^2+1}{u\sqrt{u}}}$$

$$f''_{xy} = \left(\frac{x}{\sqrt{u}}\right)'_y = \frac{\left(0\cdot\sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \underline{\frac{-xy}{u\sqrt{u}}}$$

$$f''_{yy} = \left(\frac{y}{\sqrt{u}}\right)'_y = \frac{\left(1\cdot\sqrt{u} - y \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \frac{u - y^2}{u\sqrt{u}} = \underline{\frac{x^2+1}{u\sqrt{u}}}$$

$$H(f)(0,0) = \begin{pmatrix} 1/1\sqrt{1} & 0/1\sqrt{1} \\ 0/1\sqrt{1} & 1/1\cdot\sqrt{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\left.\begin{array}{l} \det H(f)(0,0) = AC - B^2 = 1 > 0 \\ A = 1 > 0 \end{array}\right\} \Rightarrow$ $(0,0)$ is a $\underline{\text{local min}}$

d) $f'_x = 1\cdot \ln x + x\cdot \frac{1}{x} = \ln x + 1$

$f'_y = \ln y + 1$

Stat. pts:

$\ln x + 1 = \ln y + 1 = 0$

$x = y = e^{-1} \Rightarrow (x,y) = (e^{-1}, e^{-1})$

$f''_{xx} = 1/x \quad f''_{xy} = 0 \quad f''_{yy} = 1/y$

$H(f)(e^{-1},e^{-1}) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \Rightarrow (e^{-1},e^{-1})$ is local min

$AC - B^2 = e^2 > 0,\ A = e > 0$

$\circledast$ = Difficult

e) $\circledast$ $f'_x = \ln y - y\cdot\frac{1}{x} = \ln y - \frac{y}{x} = 0$

$f'_y = x\cdot\frac{1}{y} - \ln x = \frac{x}{y} - \ln x = 0$

Stat. pts:

$\ln y = \frac{y}{x} \Rightarrow x = \frac{y}{\ln y} \Rightarrow \ln\left(\frac{y}{\ln y}\right) = \frac{y/\ln y}{y} = \frac{1}{\ln y}$

$\ln x = \frac{x}{y}$

$\ln y \cdot \ln\left(\frac{y}{\ln y}\right) = 1$

$\ln y \cdot (\ln y - \ln(\ln y)) = 1$

$u(y) = \ln(y)(\ln y - \ln(\ln y))$

$u' = \frac{1}{y}(\ln y - \ln(\ln y)) + \ln y\cdot\left(\frac{1}{y} - \frac{1}{\ln y}\cdot\frac{1}{y}\right)$

$= \frac{\ln y - \ln(\ln y) + \ln y - 1}{y}$

$= \frac{2\ln y - \ln(\ln y) - 1}{y}$

To check if $u=1$ has solutions, find out when u is inc. / dec.

⇓

look at sign of u'

$u' = 0$:

$2\ln y - \ln(\ln y) = 1$

$\ln\left(\frac{y^2}{\ln y}\right) = 1$

$\frac{y^2}{\ln y} = e$

$v = \frac{y^2}{\ln y}$

$v' = \frac{2y\ln y - y^2\cdot\frac{1}{y}}{(\ln y)^2}$

$= \frac{y(2\ln y - 1)}{(\ln y)^2}$

To check if $v = e$ has solutions, find out where v is inc. / dec.

$\Rightarrow$ look at sign of v'.

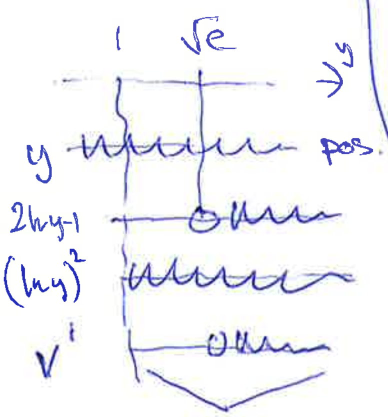
$v' = 0$:

$y\cdot(2\ln y - 1) = 0$

$2\ln y - 1 = 0$

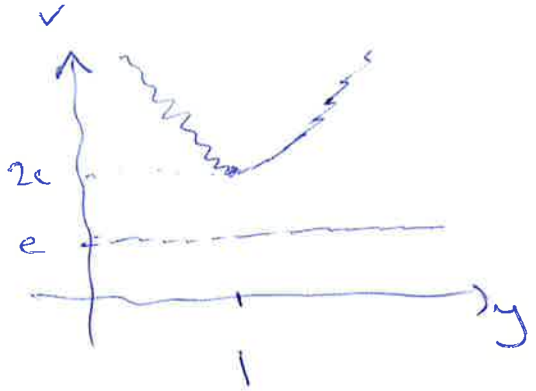
$\ln y = \frac{1}{2}$

$y = e^{1/2} = \sqrt{e}$



min for v:

$y = \sqrt{e} \Rightarrow v = \frac{e}{1/2} = 2e > e$



$v(y) = \frac{y^2}{\ln y}$

$v = e$ no solutions

$u' = 0$ no solutions

$u' = \frac{2\ln y - \ln(\ln y) - 1}{y}$

$y > 1$: $y > 0$, $2\ln y - \ln(\ln y) - 1$ const. sign since it is never zer

$y = e \rightsquigarrow 2 - \ln 1 - 1 = 1 > 0$

$\Downarrow$

$u' > 0$ for all $y > 1$

u increasing fn.

$\Downarrow$

$u = 1$ has at most one solution

u inc. function on $1 < y < \infty$

$y = e$ is a solution since $\ln e\,(\ln(e) - \ln(\ln e)) = 1 \cdot (1 - 0) = 1$

$\Downarrow$

$y = e$ only solution of $u = 1$

$x = \frac{y}{\ln y} = \frac{e}{\ln e} = e$

$\Downarrow$

$(x, y) = (e, e)$ unique stat. pt. of f.

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$

$$H(f)(e,e) = \begin{pmatrix} 1/e & 0 \\ 0 & 1/e \end{pmatrix}$$

$AC - B^2 = 1/e^2 > 0$

$A = 1/e > 0$

$\Downarrow$

$(x, y) = (e, e)$ is local min

2.

a) $\mathcal{L} = 3x + 4y - \lambda \cdot (x^2 + y^2)$

$$\text{FOC} \begin{cases} \mathcal{L}'_x = 3 - \lambda \cdot 2x = 0 & \Rightarrow x = \frac{3}{2\lambda} \\ \mathcal{L}'_y = 4 - \lambda \cdot 2y = 0 & \Rightarrow y = \frac{4}{2\lambda} \end{cases}$$

$$C \; \{ \; x^2 + y^2 = 25$$

$$x^2 + y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 25$$

$$\frac{9+16}{4\lambda^2} = 25$$

$$\frac{25}{4\lambda^2} = 25$$

$$4\lambda^2 = 1$$

$$\lambda^2 = 1/4$$

$$\underline{\lambda = \pm 1/2}$$

$\underline{\lambda = 1/2}$: $x = 3$, $y = 4$

$\Downarrow$

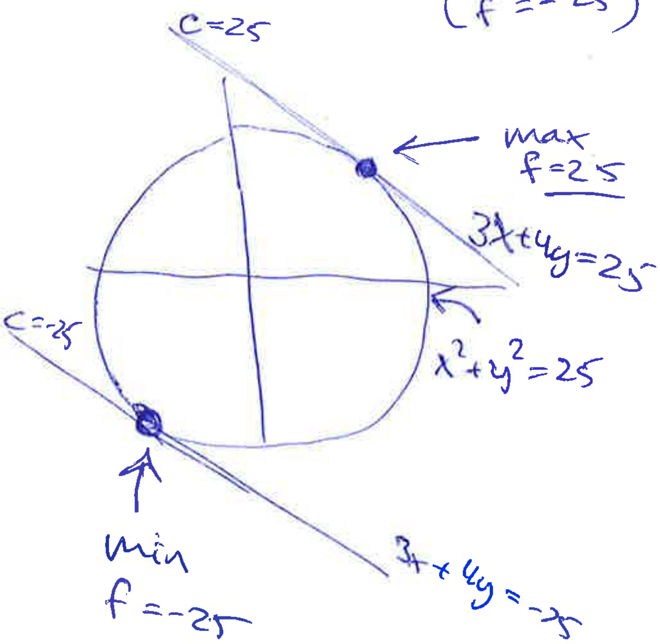
$(x, y; \lambda) = \underline{(3, 4; 1/2)}$

$(f = 25)$

$\underline{\lambda = -1/2}$: $x = -3$, $y = -4$

$(x, y; \lambda) = \underline{(-3, -4; -1/2)}$

$(f = -25)$



inc. values of c means lines = level curves move up and to the right

b) max y when $x^2+y^3=0$

$$h = y - \lambda\cdot(x^2+y^3)$$

FOC $\begin{cases} h'_x = -\lambda\cdot 2x = 0 \\ h'_y = 1-\lambda\cdot 3y^2 = 0 \end{cases}$ $\Rightarrow$ $\underline{\lambda = 0}$ or $\underline{x=0}$

$\lambda=0$: $1 = 0$, imp.

$x=0$ $\Downarrow$ $x^2+y^3=0 \Rightarrow y=0$, $1-\lambda\cdot 3y^2 = 0 \Rightarrow 1=0$, imp.

C $\{\; x^2+y^3=0$

no solution of FOC+C.

$$\boxed{\begin{aligned} g'_x &= 2x = 0 \\ g'_y &= 3y^2 = 0 \\ x^2&+y^3=0 \end{aligned}}$$

$x=0$

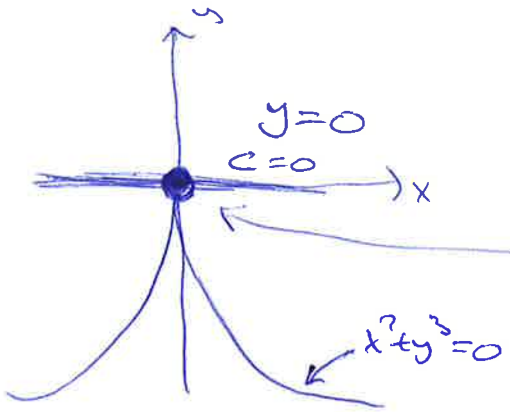
$y=0$

$x=y=0$ ok.

$(0,0)$ is adm. pt. with $g'_x = g'_y = 0$

$\Downarrow$

can be max



inc. values of c means level curve $y=c$ moves up.

$$\text{Max} = 0$$

c) min $f = 3x^2 + 4y^2$ when $xy = 1$

$$L = 3x^2 + 4y^2 - \lambda \cdot xy$$

$$\text{FOC}\begin{cases} L'_x = 6x - \lambda y = 0 \\ L'_y = 8y - \lambda x = 0 \end{cases} \qquad C\{\ xy = 1$$

(1) $x = \frac{\lambda y}{6}$

(2) $8y - \lambda \cdot \left(\frac{\lambda y}{6}\right) = 0 \quad | \cdot 6$

$48y - \lambda^2 y = 0$

$y(48 - \lambda^2) = 0$

$\underline{y = 0}$ or $\lambda^2 = 48$

$\underline{\lambda = \pm\sqrt{48}}$

$y = 0$

(3) $xy = 1$

$x \cdot 0 = 1$

$\underline{\text{imp.}}$

$\underline{\text{no soln.}}$

$\lambda = \sqrt{48}$

(1) $x = \frac{\sqrt{48}}{6} y$

(3) $xy = \frac{\sqrt{48}}{6} y \cdot y = 1$

$y^2 = \frac{6}{\sqrt{48}} = \frac{\cancel{2} \cdot 3}{\cancel{\sqrt{4}} \cdot \sqrt{12}}$

$= \frac{3}{\sqrt{12}} = \frac{\cancel{\sqrt{3}} \cdot \sqrt{3}}{\cancel{\sqrt{3}} \cdot \sqrt{4}}$

$= \sqrt{3/4}$

$y = \pm\sqrt[4]{3/4}$

$x = \pm\sqrt[4]{3/4} \cdot \frac{\sqrt{48}}{6}$

$= \pm\sqrt[4]{4/3}$

$\pm\sqrt[4]{3/4} \cdot \sqrt{\frac{48}{6}}$

$= \pm\sqrt[4]{\frac{3}{4}} \cdot \sqrt[4]{\left(\frac{4}{3}\right)^2}$

$= \pm\sqrt[4]{4/3}$

(3) $\lambda = -\sqrt{48}$

(1) $x = -\frac{\sqrt{48}}{6} y$

(3) $xy = -\frac{\sqrt{48}}{6} y \cdot y = 1$

$y^2 = -\frac{6}{\sqrt{48}}$

$\underline{\text{imp.}}$

no. $\underline{\text{soln}}$.

Pts:

$\left(\sqrt[4]{4/3}, \sqrt[4]{3/4}; \sqrt{48}\right)$

$\left(-\sqrt[4]{4/3}, -\sqrt[4]{3/4}; \sqrt{48}\right)$

$f = 3 \cdot \sqrt{4/3} + 4 \cdot \sqrt{3/4}$

$\simeq 6.93$ min pt./value

inc. values of c means the level curve = ellipse $3x^2 + 4y^2 = c$ moves outwards ("radius" increases)