Plan:

1. Determinants
2. Applications

Reading:
[ME] 9.1-9.2, 26.1-26.3

1 Determinants

$A \rightsquigarrow \det(A),\ |A|$

$n \times n$-matrix (square)

determinant of A, a number

Easy cases:

$n=1$: $A = (a)$ $\quad \det(A) = |a| = a$

$n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = ad - bc$

Ex: $A = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$ $\quad \det(A) = \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$

$= 1 \cdot (-1) - 3 \cdot 3$

$= -10$

The case $n > 2$:

Methods: for computing determinants

- crossed lines (only for $n=3$)
- cofactor expansion
- Gaussian elimination

Ex: $A = \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{array}\right) \begin{array}{cc} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{array}$

"crossed lines" **only** works for $n=3$

$$\det(A) = 1\cdot 2\cdot 9 + 1\cdot 4\cdot 1 + 1\cdot 1\cdot 3 - 1\cdot 2\cdot 1 - 3\cdot 4\cdot 1 - 9\cdot 1\cdot 1$$

$$= 18 + 4 + 3 - 2 - 12 - 9$$

$$= 25 - 23 = \underline{\underline{2}}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad |A| = aei + bfg + cdh - ceg - fha - ibd$$

Cofactor expansion:

method for computing any determinant

along a row or column

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$|A| = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$$

$$= +1 \cdot M_{11} - 1 \cdot M_{12} + 1 \cdot M_{13}$$

$$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= + (18 - 12) - (9 - 4) + (3 - 2)$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

C_{ij} = cofactor in position (i,j)

$= (-1)^{i+j} \cdot M_{ij}$

M_{ij} = minor in position (i,j)

= determinant of the submatrix you get by erasing $\begin{cases} \text{row } i \\ \text{col. } j \end{cases}$

4×4 determinant $= 4$ 3×3-det.

$= 4 \cdot 3$ 2×2-det.

$= 4 \cdot 3 \cdot 2$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c \cdot (dh - eg)$$

$$\begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & & \\ - & + & & & \\ + & - & & \ddots & \end{pmatrix}$$

$$\begin{pmatrix} \cancel{\;} & \cancel{\;} & \cancel{\;} \\ \cancel{\;} & 2 & 4 \\ \cancel{\;} & 3 & 9 \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 2 \cdot 9 - 3 \cdot 4 = 6$$

Determinants can be computed by cofactor expansion along any row or column (with the same result).

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c \cdot (dh - eg)$$

$$|A| = -d(bi - ch) + e(ai - cg) - f(ah - bg)$$

note that these expressions are equal.

Ex:

$$\begin{vmatrix} 0 & 7 & 3 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 4 & -1 & 2 \end{vmatrix} = 0 \cdot * - 1 \cdot \begin{vmatrix} 7 & 3 & 4 \\ 1 & 0 & 1 \\ 4 & -1 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 7 & 3 & 4 \\ 0 & -1 & 2 \\ 4 & -1 & 2 \end{vmatrix} - 0 \cdot *$$

$$= -\left(-1 \cdot \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 7 & 3 \\ 4 & -1 \end{vmatrix}\right)$$

$$+ 2 \cdot \left(-1 \cdot \begin{vmatrix} 7 & 4 \\ 4 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 7 & 3 \\ 4 & -1 \end{vmatrix}\right)$$

$$= +10 + (-19) - 2 \cdot (-2) - 4 \cdot (-19)$$

$$= 10 - 19 + 4 + 76 = \underline{\underline{71}}$$

Special cases:

① Diagonal matrices:

$$A = \begin{pmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

$$\boxed{|A| = d_1 \cdot d_2 \cdot \ldots \cdot d_n \text{ for any diagonal matrix}}$$

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$|A| = 1 \cdot \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= \underline{1 \cdot (-1) \cdot 3}$$

$$|A| = 1 \cdot (-1) \cdot 3$$

$$= \underline{\underline{-3}}$$

② Upper triangular matrices

$$A = \begin{pmatrix} d_1 & * & * & \cdots \\ 0 & d_2 & * & \cdots \\ 0 & 0 & d_2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$|A| = d_1 \cdot d_2 \cdot \ldots \cdot d_n$

for any upper triangular matrix A

Ex:

$$\begin{vmatrix} 1 & 7 & 4 & \sqrt{3} \\ 0 & 2 & -17 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

Any echelon form is upper triangular

$$\# = 1 \cdot \begin{vmatrix} 2 & -17 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 1 \cdot \left(2 \cdot \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix}\right)$$

$$= 1 \cdot 2 \cdot (-1) \cdot 2 = \underline{\underline{-4}}$$

Method: Gaussian elimination

Ex: $A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 0 & 4 \\ 0 & 7 & -1 & 3 \\ 1 & 2 & 1 & 4 \end{pmatrix}$ -2 -1

$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 7 & -1 & 3 \\ 0 & -1 & 2 & 2 \end{pmatrix} = B$

$|A| = |B| =$

$= 1 \cdot \begin{vmatrix} -5 & 2 & 0 \\ 7 & -1 & 3 \\ -1 & 2 & 2 \end{vmatrix}$

$= -5 \cdot (-2-6)$
$-2\,(14+3)$
$= 40 - 34 = \underline{\underline{6}}$

$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 7 & -1 & 3 \\ 0 & -5 & 2 & 0 \end{pmatrix}$ 7 -5

$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 13 & 17 \\ 0 & 0 & -8 & -10 \end{pmatrix}$ $8/13$

$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 13 & 17 \\ 0 & 0 & 0 & -10 + \frac{8 \cdot 17}{13} \end{pmatrix} = E$

$|E| = 1 \cdot (-1) \cdot 13 \cdot \left(-10 + \frac{8 \cdot 17}{13}\right) = 130 - 136 = \underline{-6}$

Fact: elementary row operation: $A \to B$

1. switch two rows $|B| = -|A|$
2. multiply a row by $c \neq 0$ $|B| = c \cdot |A|$
3. add a multiple of a row to another row $|B| = |A|$

$\underbrace{A \to \to \to \ldots}_{\text{all of type 3, except one row interchange (1)}} \to E$

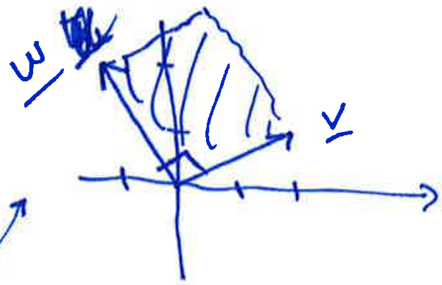
$|E| = -6$

$|A| = -|E| = \underline{\underline{6}}$

2) Applications

Geometric interpretation:

Ex: $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$



$|A|$ = area of the parallelogram

$B = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$

$|B| = -1 - 4 = -5$

$= -$ area

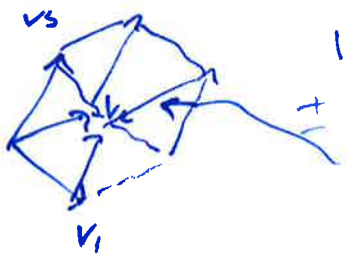
$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5$

$|\underline{v}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$|\underline{w}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$



$|A| =$
$\pm$ volume of this

$\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$

Determinants and inverse matrices:

A $n\times n$-matrix

$\boxed{A^{-1} \text{ exists if and only if } |A| \neq 0}$

In this case, we have

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$$
$$= \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$$

Ex:
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$|A| = 1\cdot(2\cdot 9 - 3\cdot 4) - 1\cdot(9-4) + 1\cdot(3-2)$$
$$= 6 - 5 + 1 = \underline{2} \neq 0$$

$\Downarrow$

$C_{11} = 6$ $C_{12} = -5$ $C_{13} = 1$

$C_{21} = -6$ $C_{22} = +8$ $C_{23} =$

$C_{31} =$ $C_{32} =$ $C_{33} =$

A^{-1} exists, and

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}^T$$
$$= \frac{1}{2}\begin{pmatrix} 6 & 6 & \cdot \\ -5 & 8 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$$

Determinants and linear systems:

$$\boxed{\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}} \longrightarrow \quad A \cdot \underline{x} = \underline{b}$$

$m \times n$ linear system

matrix form

A: $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

In case $m = n$:

$|A|$ gives a lot of information about the linear system

- $\underline{|A| \neq 0}$: exactly one solution
- $\underline{|A| = 0}$: either no sol'ns or infinitely many solutions

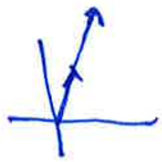
Ex:

$$x + 2y = 3$$
$$2x + 4y = 6$$

infinitely many solutions

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$|A| = 0$$



$$x + 2y = 3$$
$$2x + 4y = 5$$

no solutions

$$x + 2y = 3$$
$$2x - y = 5$$

one solution

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

Explanation:

$|A| \neq 0$: one solution

$|A| = 0$: no sol'n or inf. many sol'n.

$|A| \neq 0 \Rightarrow A^{-1}$ exists :

$$A\underline{x} = \underline{b}$$
$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$

one sol'n for all values of $\underline{b}$.

$|A| = 0 \Rightarrow A^{-1}$ does not exist

Alternative:

$$(A \mid \underline{b}) \to \dots \to (E \mid \underline{c})$$

echelon form

$$\left(\begin{array}{ccc|c} 1 & \cdot & \cdot & \cdot \\ 0 & 2 & \cdot & \cdot \\ 0 & 0 & 3 & \cdot \end{array}\right)$$

one solution

$|A| \neq 0$
$\Uparrow$
$|E| \neq 0$

$$\left(\begin{array}{c|c} \boxed{\begin{array}{ccc} 1 & \cdot & \cdot \\ 0 & 2 & \cdot \\ 0 & 0 & 0 \end{array}} & \begin{array}{c} \cdot \\ \cdot \\ 0 \end{array} \end{array}\right)$$

inf. many solutions

$|A| = 0$

$$\left(\begin{array}{c|c} \boxed{\begin{array}{ccc} 1 & \cdot & \cdot \\ 0 & 2 & \cdot \\ 0 & 0 & 0 \end{array}} & \begin{array}{c} \cdot \\ \cdot \\ * \end{array} \end{array}\right) \quad * \neq 0$$

no sol'n

$|A| = 0$

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Computing the determinant of a 3×3 *matrix.*

Figure 9.1

Theorem 9.3 A square matrix is nonsingular if and only if its determinant is nonzero.

Proof Sketch Recall that a square matrix A is nonsingular if and only if its row echelon form R has no all-zero rows. Since each row of the square matrix R has more leading zeros than the previous row, R has no all-zero rows if and only if the jth row of R has exactly $(j - 1)$ leading zeros. This occurs if and only if R has no zeros on its diagonal. Since $\det R$ is the product of its diagonal entries, A is nonsingular if and only if $\det R$ is nonzero. Since $\det R = \pm \det A$, A is nonsingular if and only if $\det A$ is nonzero. ■

Theorem 9.3 is obvious for 1×1 matrices, because the equation $ax = b$ has a unique solution, $x = b/a$, for every b if and only if $a \neq 0$. Theorem 8.8 demonstrates Theorem 9.3 for 2×2 matrices.

EXERCISES

9.1 Write out the complete expression for the determinant of a 3×3 matrix — six terms, each a product of three entries.

9.2 Write out the definition of the determinant of a 4×4 matrix in terms of the determinants of certain of its 3×3 submatrices. How many terms are there in the complete expansion of the determinant of a 4×4 matrix?

9.3 Compute out the expression on the right-hand side of (5). Show that it equals the expression calculated in Exercise 9.1.

9.4 Show that one obtains the same formula for the determinant of a 2×2 matrix, no matter which row or column one uses for the expansion.

9.5 Use a formula for the determinant to verify Theorem 9.1 for upper-triangular 3×3 matrices.

9.6 Verify the conclusion of Theorem 9.2 for 2×2 matrices by showing that the determinant of a general 2×2 matrix is not changed if one adds r times row 1 to row 2.

9.7 For each of the following matrices, compute the row echelon form and verify the conclusion of Theorem 9.2:

$$a)\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}.$$

9.8 Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}, \qquad b) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

9.9 Use Theorem 9.3 to determine which of the matrices in Exercises 9.7 and 9.8 are nonsingular.

9.2 USES OF THE DETERMINANT

Since the determinant tells whether or not A^{-1} exists and whether or not $A\mathbf{x} = \mathbf{b}$ has a unique solution, it is not surprising that one can use the determinant to derive a formula for A^{-1} and a formula for the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. First, we define the adjoint matrix of A as the transpose of the matrix of cofactors of A.

Definition For any $n \times n$ matrix A, let C_{ij} denote the (i, j)th cofactor of A, that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j from A. The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of A (note the switch in indices), is called the **adjoint** of A and is written **adj** A.

Theorem 9.4 Let A be a nonsingular matrix. Then,

(a) $A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$, and

(b) (**Cramer's rule**) the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \qquad \text{for } i = 1, \ldots, n,$$

where B_i is the matrix A with the right-hand side $\mathbf{b}$ replacing the ith column of A.

For 3×3 systems,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Finally, we note three algebraic properties of the determinant function which we will find important in our use of determinants.

Theorem 9.5 Let A be a square matrix. Then,

(*a*) $\det A^T = \det A$,
(*b*) $\det(A \cdot B) = (\det A)(\det B)$, and
(*c*) $\det(A + B) \neq \det A + \det B$, in general.

Gaussian elimination is a much more efficient method of solving a system of n equations in n unknowns than is Cramer's rule. Cramer's rule requires the evaluation of $(n+1)$ determinants. Each determinant is a sum of $n!$ terms and each term is a product of n entries. So, Cramer's rule requires $(n+1)!$ operations. On the other hand, the number of arithmetic operations required by Gaussian elimination for such a system is on the order of n^3. If $n = 6$ as in the Leontief model in Section 8.5, then $(n+1)!$ is 5040, while n^3 is only 216; the difference grows exponentially as n increases.

Nevertheless, Cramer's rule is particularly useful for small linear systems in which the coefficients a_{ij} are parameters and for which one wants to obtain a general formula for the endogenous variables (the x_i's) in terms of the parameters and the exogenous variables (the b_j's). One can then see more clearly how changes in the parameters affect the values of the endogenous variables.

EXERCISES

9.10 Verify directly that matrix (9) really is the inverse of matrix (8) in Example 9.3.

9.11 Use Theorem 9.4 to invert the following matrices:

$$a)\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}, \qquad c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

9.12 Use Cramer's rule to compute x_1 and x_2 in Example 9.4.

9.13 Use Cramer's rule to solve the following systems of equations:

$$a)\ \begin{array}{l} 5x_1 + x_2 = 3 \\ 2x_1 - x_2 = 4; \end{array} \qquad b)\ \begin{array}{rl} 2x_1 - 3x_2 & = 2 \\ 4x_1 - 6x_2 + x_3 & = 7 \\ x_1 + 10x_2 & = 1. \end{array}$$

9.14 Verify the conclusions of Theorem 9.5 for the following pairs of matrices:

$$a)\ A = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix};$$

EXERCISES

26.1 Compute the determinant of each of the following matrices:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\quad b)\ \begin{pmatrix} 4 & 2 \\ -8 & -4 \end{pmatrix},\quad c)\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix},\quad e)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 0 & 3 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

26.2 Calculate $\det A$ for the last three matrices in the previous exercise by expanding along a column of A and along a row other than the first. Note that for the last two matrices, expanding along a row with many zeros can simplify the calculation.

26.3 Show that if $a_{11}a_{22} - a_{21}a_{12} = 0$, then the expression for $\det A$ in (5) equals $-(a_{23}a_{11} - a_{21}a_{13})(a_{11}a_{32} - a_{31}a_{12})/a_{11}$.

26.4 *a*) Check that (10) yields the same formula as (5).

b) Use another row and another column of the 3×3 matrix A to calculate $\det A$ and check that you obtain the same expression as (5).

26.5 Write out a careful proof of Theorem 26.2.

26.6 Use the method of Figure 26.1 to compute the two 3×3 determinants in Exercise 26.1.

26.7 *a*) How many terms are there in the formula for the determinant of a general $n \times n$ matrix?

b) How many arithmetic operations (additions, subtractions, etc.) are needed to compute the determinant of a general $n \times n$ matrix?

26.8 If one used the technique described in Figure 26.1, the "determinant" of a 4×4 matrix would require only eight terms. Compare this with the number of terms that are indicated in the previous exercise for a 4×4 determinant.

26.2 PROPERTIES OF THE DETERMINANT

We still must show that formula (8) for the determinant really works — that a matrix is nonsingular if and only if its determinant is nonzero. This result is the goal of this section. Along the way we will develop some properties of the determinant. Since we are collecting all the important facts about the determinant in this section, we will begin by repeating the important fact which we proved at the end of the last section — Theorem 26.2.

Fact 26.1. For any $n \times n$ matrix A, $\det A = \det A^T$.

Fact 26.1 implies that any statement about how the rows of a matrix affect the value of the determinant is also true when applied to the columns of a matrix. We

Theorem 26.4 can be applied to the equation $AA^{-1} = I$ to compute $\det A^{-1}$ in terms of $\det A$.

Theorem 26.5 If A is invertible, $\det A^{-1} = 1/\det A$.

Example 26.6 If $A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$. It is easy to compute that $\det A = 2$ and $\det A^{-1} = 1/2 = 1/\det A$.

EXERCISES

26.9 Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 6 & 4 \\ 1 & 1 \end{pmatrix}$.

a) Show that $\det(A + B) \neq \det A + \det B$.

b) Show that $\det A + \det B = \det C$ and relate this to Fact 26.6.

26.10 Use induction to supply a more careful proof of Fact 26.8.

26.11 Write out a careful proof of Fact 26.11.

26.12 Show that an upper- or lower-triangular matrix is nonsingular if and only if every diagonal entry is nonzero.

26.13 *a)* Compute the determinant of each of the following matrices by applying row operations to obtain an upper-triangular matrix and then use Fact 26.11:

$$i)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix}, \quad ii)\ \begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 4 \\ 4 & 6 & 2 & 0 \end{pmatrix}, \quad iii)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & 12 & 2 \end{pmatrix}.$$

b) Which of these matrices are nonsingular?

26.14 Find the exact values of k which make each of the following matrices singular:

$$a)\ \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}.$$

26.15 Prove Theorem 26.5.

26.16 Prove the following results for $n \times n$ matrices:

a) $\det rA = r^n \cdot \det A$;

b) $\det(-A) = (-1)^n \det A$;

c) $\det(A_1 \cdots A_r) = (\det A_1) \cdots (\det A_r)$;

d) $\det A^k = (\det A)^k$ for positive integers k;

e) $\det A^k = (\det A)^k$ for all integers k if A is invertible.

26.17 Finish the proof of Lemma 26.1 for the case $A = E_{ij}(r)$.

26.18 *a)* An orthogonal matrix is a nonsingular matrix such that $A^{-1} = A^T$. Show that the determinant of an orthogonal matrix is ± 1.

b) A skew symmetric matrix is a square matrix such that $A^T = -A$. Show that if n is odd, a skew symmetric matrix is singular.

c) Present some nontrivial examples of orthogonal matrices and skew-symmetric matrices.

26.19 Show that the determinant of A is, up to sign, the product of its pivots.

26.20 Show that two $n \times n$ matrices A and B are invertible (nonsingular) if and only if their product AB is invertible (nonsingular).

26.21 (difficult) Suppose that you are given a square matrix A, partitioned into four submatrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} and A_{22} are square submatrices.

a) Show that

$$\det \begin{pmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{pmatrix} = \det A_{11} \cdot \det A_{22}.$$

b) Show that

$$\det \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det \begin{pmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{pmatrix} = \det A_{11} \cdot \det A_{22}.$$

c) Suppose that A_{22} is nonsingular. Show that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ \mathbf{0} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} I & \mathbf{0} \\ A_{22}^{-1}A_{21} & I \end{pmatrix}.$$

d) Conclude that if A_{22} is nonsingular,

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \cdot \det A_{22}.$$

e) Use this method to compute

$$\det \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 4 & 1 \\ -3 & 1 & 3 & 1 \\ 4 & 2 & 5 & 2 \end{pmatrix}.$$

26.22 Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - a)(c - a)(c - b).$$

26.3 USING DETERMINANTS

The discussion in the last section showed that the determinant as defined in (6) is an effective tool for checking whether or not a square matrix is nonsingular. In this section, we will describe some other applications of the determinant.