Plan:

1. Introduction
2. Solving linear systems
3. Gaussian elimination

Readings:

[ME] 6.1, 7.1-7.3

1 Introduction

FORK 1003 — mathematics (6 lectures)

MSc Bus./Fin. — economics

— econometrics

GRA 6035 Mathematics

- matrices
- calculus

Textbook: [ME] Simon, Blume: Mathematics for economists

① Linear systems

a) $7x - 3y = 6$
$2x + y = 11$

b) $x - y - 2z + 3w = 2$
$2x - y + z + 5w = 3$
$x + 7y + 10z + 5w = 4$

Methods for linear systems

- substitution methods
- elimination methods

Ex: a) $7x - 3y = 6$
$2x + y = 11 \longrightarrow y = 11 - 2x$

substitute

$$7x - 3y = 6$$
$$7x - 3(11 - 2x) = 6$$
$$7x - 33 + 6x = 6$$
$$13x = 39 \quad | : 13$$
$$x = 3 \qquad y = 5$$

Solution: $(x, y) = (3, 5)$

For 2×2 linear systems:

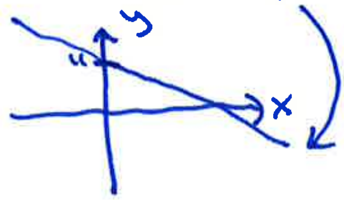
$$\boxed{\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}}$$

a, b, c, d, e, f are given numbers

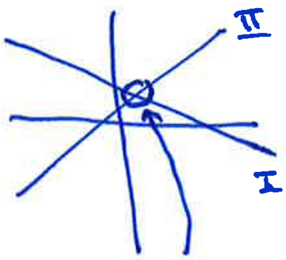
geometrically, each equation is a straight line in the xy coordinate system

Ex: $7x - 3y = 6$

$2x + y = 11 \longrightarrow y = -2x + 11$

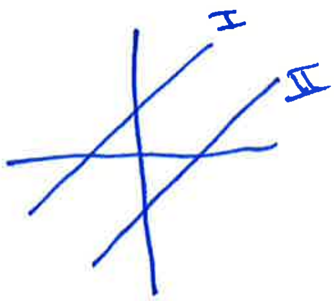


There are three cases:



one solution

consistent

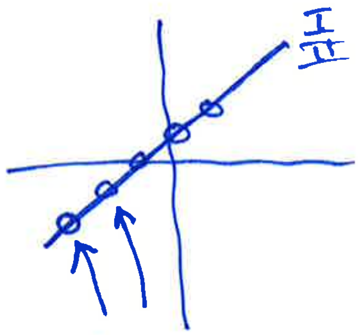


no solution

$x + 2y = 4$

$2x + 4y = 7$

inconsistent



infinitely many solutions

$x + 2y = 4$

$2x + 4y = 8$

consistent

For any $m \times n$ linear system $\left(\begin{array}{l} m \text{ equations} \\ n \text{ variables} \end{array}\right)$, exactly one of the following cases occur:

i) there is exactly <u>one solution</u>

ii) there are <u>no solutions</u>

iii) there are <u>infinitely many solutions</u>

We cannot have something like this: (in a <u>linear</u> system)

<u>Elimination method:</u>

$R(2) := R(2) + 1 \cdot R(1)$

a) $\begin{aligned} 7x - 3y &= 6 \\ 2x + y &= 11 \quad | \cdot 3 \end{aligned} \rightarrow \begin{aligned} 7x - 3y &= 6 \\ 6x + 3y &= 33 \end{aligned} \quad 1\cdot \rightarrow$

$13x = 39$

$\rightarrow \begin{aligned} 7x - 3y &= 6 \\ 13x &= 39 \end{aligned} \rightarrow \begin{aligned} &7 \cdot 3 - 3y = 6 \\ &\underline{x = 3} \end{aligned} \quad \frac{-3y}{-3} = 6 - 21 = \frac{-15}{-3}$

$\underline{y = 5}$

backwards substitution

<u>Matrix notation:</u>

$$\left(\begin{array}{cc|c} 7 & -3 & 6 \\ 2 & 1 & 11 \end{array}\right)$$

<u>Solution</u>: $(x, y) = \underline{\underline{(3,5)}}$

Definition: A linear equation in $x_1, x_2, \ldots, x_n$ has the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

where $a_1, a_2, \ldots, a_n, b$ are given numbers.

Ex: $2x_1 - x_2 + x_3 = 2$ is linear

$x_1 + x_2^2 = 4$ is not linear

$x_1 x_2 = 1$ — " —

An $m \times n$ linear system is a system of m linear equations in n variables $x_1, x_2, \ldots, x_n$. It has the form

a_{12}: coeff. in front of x_2 in the first eqn.

$$m \left\{ \begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \\ \underbrace{a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n}_{n} = b_m \end{array} \right.$$

where $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$ are given numbers.

The coefficient matrix of this linear system

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$m \times n$ matrix (m rows, n col's)

The augmented matrix is

$$\hat{A} = (A|\underline{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

② Gaussian elimination : systematic method for solving any linear system, elimination method, efficient, give good understanding

$\boxed{\text{linear system}} \longrightarrow \boxed{\text{augmented matrix}}$

$\downarrow$ elementary row operations

$\boxed{\text{linear system with var. elim.}} \longleftarrow \boxed{\text{echelon form}}$

$\downarrow$ backwords substitution

$\boxed{\text{solutions}}$

Echelon form:

Ex:

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & 4 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right) \longleftrightarrow \begin{cases} x+3y-z+2w=0 \\ 4y+z=3 \\ w=7 \end{cases}$$

augmented matrix in echelon form

lin. system.

In general: The first (left-most) non-zero entry in a row is called a pivot (circled)

A matrix is in echelon form if:

i) all zero rows (rows with only zeros) are below all other rows

ii) all entries below a pivot are zero.

Ex:
$$\begin{aligned} x+3y-z+2w&=0\\ 4y+z&=3\\ w&=7 \end{aligned} \iff \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0\\ 0 & 4 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

echelon form

Backwards substitution:

III $w=7$

II $4y+z=3 \to \frac{4y}{4}=\frac{3-z}{4} \Rightarrow y=\frac{3}{4}-\frac{1}{4}z$

I $x+3y-z+2w=0$

$$x=-3y+z-2w$$
$$=-3\cdot\left(\frac{3}{4}-\frac{1}{4}z\right)+z-2\cdot 7$$
$$=-\frac{9}{4}+\frac{3}{4}z+z-14=\frac{7}{4}z-\frac{65}{4}$$

Solution: $(x,y,z,w)=\left(\frac{7}{4}z-\frac{65}{4},\ \frac{3}{4}-\frac{1}{4}z,\ z,\ 7\right)$

z is called a free variable

(infinitely many solutions)

Ex:

$$x + y + z = 2$$
$$x + 2y + 4z = 1$$
$$x + 3y + 9z = -2$$

$R(2) := R(2) + (-1) \cdot R(1)$

not echelon form

$\neq 0$ need to make them zero.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & -2 \end{array}\right) \begin{array}{c} \\ -1 \\ \\ \end{array}$$

Elementary row operat.:

1. interchange two rows
2. multiply a row by $c \neq 0$
3. add a multiple of one row to another row

$$\begin{array}{c} \\ \\ -1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 1 & 3 & 9 & -2 \end{array}\right)$$

$$\begin{array}{rcccc|c} & R(2) & 1 & 2 & 4 & 1 \\ -1\cdot & R(1) & -1 & -1 & -1 & -2 \\ \hline & & 0 & 1 & 3 & -1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 8 & -4 \end{array}\right) \begin{array}{c} \\ -2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

echelon form

Fact:
- Elementary row operations are 'legal operations', i.e. they don't change the solutions of the system
- Any (augmented) matrix can be transformed into an echelon form using elementary row operations.
- The echelon form is not unique.

  The pivot positions = pivots in an echelon form. are unique.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right) \cdot \tfrac{1}{2} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

echelon form — echelon form

$$x + y + z = 2$$
$$y + 3z = -1$$
$$z = -1$$

$z = -1$

$y = -1 - 3z = -1 + 3 = 2$

$x = 2 - y - z = 2 - 2 + 1 = 1$

Sol: $(x, y, z) = (1, 2, -1)$ one solution

Fact: You can tell how many solutions there are from pivot positions.

inconsistent:
no solutions $\Leftrightarrow$ pivot position in the last column. (after the vertical line)

one unique solution $\Leftrightarrow$ pivot position in all col's (except the last) - no free var's

infinitely many solutions $\Leftrightarrow$ not pivot position in all col's (except the last) - there are free var's

Ex:

$$\left(\begin{array}{ccc|c} 4 & 1 & -1 & 3 \\ 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right) \Leftrightarrow \begin{array}{c} \vdots \\ 0\cdot x + 0\cdot y + 0\cdot z = 2 \end{array}$$

echelon form w/ pivot positions

no sol'ns

Ex:

$$\left(\begin{array}{ccc|c} 4 & 1 & -1 & 3 \\ 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{r} \underline{4x} + y - z = 3 \\ \underline{7y} + z = 4 \end{array}$$

echelon form

$\underbrace{\hphantom{xxxxxxxxxxxx}}_{\text{variable col's}}$ $\left\{ \begin{array}{l} x, y: \text{ basic} \\ z: \text{ free} \end{array} \right.$

infinitely many solutions

Problems: [ME] 7.1-7.3, 7.12-7.16

Note: Gauss-Jordan elimination is a variation of Gaussian elim. where you use row operations until you get a

reduced echelon form

= echelon form where, in addition, all pivots are 1 and also entries over a pivot are zero.

You may use ordinary Gaussian elimination in the problems instead of Gauss-Jordan elimination.

nonzero coefficient is 1:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3\ x_3 &= 130 \\ x_2 - 0.25x_3 &= 125 \\ x_3 &= 300. \end{aligned} \tag{11}$$

Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11). For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of x_3 in (11b) and obtain $x_2 = 200$. Then, add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$\begin{aligned} x_1 \qquad\qquad &= 300 \\ x_2 \qquad &= 200 \\ x_3 &= 300, \end{aligned} \tag{12}$$

which needs no further work to see the solution. Gauss-Jordan elimination is particularly useful in developing the theory of linear systems; Gaussian elimination is usually more efficient in solving actual linear systems.

Earlier we mentioned a third method for solving linear systems, namely matrix methods. We will study these methods in the next two chapters, when we discuss matrix inversion and Cramer's rule. For now, it suffices to note that all the intuition behind these more advanced methods derives from Gaussian elimination. The understanding of this technique will provide a solid base on which to build your knowledge of linear algebra.

EXERCISES

7.1 Which of the following equations are linear?

a) $3x_1 - 4x_2 + 5x_3 = 6$; *b)* $x_1x_2x_3 = -2$; *c)* $x^2 + 6y = 1$;
d) $(x + y)(x - z) = -7$; *e)* $x + 3^{1/2}z = 4$; *f)* $x + 3z^{1/2} = -4$.

7.2 Solve the following systems by substitution, Gaussian elimination, and Gauss-Jordan elimination:

a)
$$\begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1; \end{aligned}$$

b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 12x_1 + 2x_2 - 3x_3 &= 5 \\ 3x_1 + 4x_2 + x_3 &= -4. \end{aligned}$$

7.3 Solve the following systems by Gauss-Jordan elimination. Note that the third system requires an equation interchange.

a) $\begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned}$ *b)* $\begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned}$ c) $\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$

7.4 Formalize the three elementary equation operations using the abstract notation of system (2), and for each operation, write out the operation which reverses its effect.

7.5 Solve the IS-LM system in Exercise 6.7 by substitution.

7.6 Consider the general IS-LM model with no fiscal policy in Chapter 6. Suppose that $M_s = M^o$; that is, the intercept of the LM-curve is 0.

a) Use substitution to solve this system for Y and r in terms of the other parameters.

b) How does the equilibrium GNP depend on the marginal propensity to save?

c) How does the equilibrium interest rate depend on the marginal propensity to save?

7.7 Use Gaussian elimination to solve

$$\begin{cases} 3x + 3y = 4 \\ -x - y = 10. \end{cases}$$

What happens and why?

7.8 Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

What assumptions do you have to make about the coefficients a_{ij} in order to find a solution?

7.2 ELEMENTARY ROW OPERATIONS

The focus of our concern in the last section was on the coefficients a_{ij} and b_i of the systems with which we worked. In fact, it was a little inefficient to rewrite the x_i's, the plus signs, and the equal signs each time we transformed a system. It makes sense to simplify the representation of linear system (2) by writing two rectangular arrays of its coefficients, called **matrices**. The first array is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

7.11 Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

7.12 Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x \quad\quad - z &= 0. \end{aligned}$$

7.3 SYSTEMS WITH MANY OR NO SOLUTIONS

As we will study in more detail later, the locus of all points (x_1, x_2) which satisfy the linear equation $a_{11}x_1 + a_{12}x_2 = b_1$ is a straight line in the plane. Therefore, the solution (x_1, x_2) of the two linear equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16}$$

is a point which lies on both lines of (16) in the Cartesian plane. Solving system (16) is equivalent to finding where the two lines given by (16) cross. In general, two lines in the plane will be nonparallel and will cross in exactly one point. However, the lines given by (16) can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide, every point on either line is a solution to (16); and (16) has *infinitely* many solutions. An example is the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

In the case where the two parallel lines do not cross, the corresponding system has *no* solution, as the example

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

illustrates. Therefore, it follows from geometric considerations that two linear equations in two unknowns can have one solution, no solution, or infinitely many solutions. We will see later in this chapter that this principle holds for every system of *m linear* equations in *n* unknowns.

$$\begin{pmatrix} * & w & w & w & w & w & w & | & w \\ 0 & 0 & 0 & * & w & w & w & | & w \\ 0 & 0 & 0 & 0 & * & w & w & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & * & | & w \end{pmatrix}.$$

This matrix is in row echelon form. The corresponding reduced row echelon form is

$$\begin{pmatrix} 1 & w & w & 0 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 1 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 1 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & | & w \end{pmatrix}.$$

The final solution will have the form

$$\begin{aligned} x_1 &= a_1 - a_2x_2 - a_3x_3 - a_4x_6, \\ x_4 &= b_1 \qquad\qquad\qquad\quad - b_2x_6, \\ x_5 &= c_1 \qquad\qquad\qquad\quad - c_2x_6, \\ x_7 &= d_1. \end{aligned}$$

Here x_7 is the only variable which is unambiguously determined. The variables x_2, x_3, and x_6 are free to take on any values; once values have been selected for these three variables, then values for x_1, x_4, and x_5 are automatically determined.

Some more vocabulary is helpful here. If the jth column of the row echelon matrix $\hat{B}$ contains a pivot, we call x_j a **basic variable**. If the jth column of $\hat{B}$ does not contain a pivot, we call x_j a **free** or **nonbasic variable**. In this terminology, Gauss-Jordan elimination determines a solution of the system in which each basic variable is either unambiguously determined or a linear expression of the free variables. The free variables are free to take on any value. Once one chooses values for the free variables, values for the basic variables are determined.

As in the example above, the free variables are often placed on the right-hand side of the equations to emphasize that their values are not determined by the system; rather, they act as parameters in determining values for the basic variables.

In a given problem which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process and on the order in which the variables are indexed.

EXERCISES

7.13 Reduce the following matrices to row echelon and reduced row echelon forms:

a) $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, *b)* $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}$, *c)* $\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$.

7.14 Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

7.15 Use Gauss-Jordan elimination to determine for what values of the parameter k the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

7.16 Use Gauss-Jordan elimination to solve the following four systems of linear equations. Which variables are free and which are basic in each solution?

$$a)\ \begin{aligned} w + 2x + y - z &= 1 \\ 3w - x - y + 2z &= 3 \\ -x + y - z &= 1 \\ 2w + 3x + 3y - 3z &= 3; \end{aligned} \qquad b)\ \begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6 \\ 3w + 2x + 5y - z &= 3; \end{aligned}$$

$$c)\ \begin{aligned} w + 2x + 3y - z &= 1 \\ -w + x + 2y + 3z &= 2 \\ 3w - x + y + 2z &= 2 \\ 2w + 3x - y + z &= 1; \end{aligned} \qquad d)\ \begin{aligned} w + x - y + 2z &= 3 \\ 2w + 2x - 2y + 4z &= 6 \\ -3w - 3x + 3y - 6z &= -9 \\ -2w - 2x + 2y - 4z &= -6. \end{aligned}$$

7.17 *a*) Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

b) Suppose you hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

7.18 For what values of the parameter a does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

7.19 From Chapter 6, the stationary distribution in the Markov model of unemployment satisfies the linear system

$$\begin{aligned} (q - 1)x + py &= 0 \\ (1 - q)x - py &= 0 \\ x + y &= 1. \end{aligned}$$