Key Problems

Problem 1.
Show that the matrix A is a Hermitian matrix, and find a unitary matrix U such that $U^*AU = D$ is diagonal when

$$A = \begin{pmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{pmatrix}$$

Problem 2.
Use Python to find the eigenvalues and eigenvectors of the following matrices. Then check whether the matrices are irreducible, and if so, find the dominant eigenvector λ and a positive eigenvector $\mathbf{v}$:

a) `A = np.array([[0,0,1/3,1/2],[1/2,0,1/3,1/2],[1/2,1,0,0], [0,0,1/3,0]])`

b) `B = np.array([[0,3,0,0,1,2],[3,0,2,2,2,1],[6,4,0,2,1,1],` [3,1,1,0,2,2], `[2,1,2,4,0,2],[1,2,2,4,4,0]])`

Problem 3.
Write a Python program the defines a function `Frobenius(matrix)` that has a square non-negative irreducible matrix as input, and a positive eigenvector as output. Check that the function works by using it on the following matrices:

a) `A = np.array([[0,0,1/3,1/2],[1/2,0,1/3,1/2],[1/2,1,0,0], [0,0,1/3,0]])`

b) `B = np.array([[0,3,0,0,1,2],[3,0,2,2,2,1],[6,4,0,2,1,1],` [3,1,1,0,2,2], `[2,1,2,4,0,2],[1,2,2,4,4,0]])`

Problem 4.
We consider an $n \times n$ Leslie matrix A given by

$$A = \begin{pmatrix} f_1 & f_2 & f_3 & \dots & f_{n-1} & f_n \\ p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{pmatrix}$$

where $0 < p_i < 1$ and $f_i \geq 0$ for all i, and $f_n > 0$.

a) Find the characteristic equation of A in the case $n = 4$.

b) Explain why the condition $f_n > 0$ is necessary for A to be an irreducible non-negative matrix.

Problem 5.
A population of *rattus norvegicus* is divided into $n = 21$ age groups (time measured in months), and the growth of the population is given by the Leslie matrix A with parameters

```
f = np.array([0,0,0.3964,1.4939,2.1777,2.5250,2.6282, 2.6749,2.6018,2.4419,2.1865,1.9044,
              1.7259,1.4918,1.2415,0.9522,0.7141,0.4618,0.2518, 0.0901,0.0035]
p = np.array([0.94697,0.99665,0.99926,0.99899,0.99863, 0.99817,0.99753,0.99667,0.99553,
              0.99399,0.99196,0.98926,0.98572,0.98107,0.97511, 0.96748, 0.95797,0.94631,
              0.93247,0.91649])
```

a) Write a python program that defines a function `Leslie(f,p)` that has one-dimensional arrays f and p of size n and $n-1$ as inputs, and the corresponding Leslie matrix A as output.

b) Find the dominant eigenvalue and a Frobenius vector of A.

c) Plot the population growth of all age groups of this population when the initial population has 5 inviduals in each age group.

Answers to Key Problems

Problem 1.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad U = \begin{pmatrix} -1/\sqrt{7} & (1-21i)/\sqrt{728} & (1+3i)/\sqrt{40} \\ (1+2i)/\sqrt{7} & (6-9i)/\sqrt{728} & (-2-i)/\sqrt{40} \\ 1/\sqrt{7} & 13/\sqrt{728} & 5/\sqrt{40} \end{pmatrix}$$

Problem 2.
Both matrices are irreducible, and we find

a) $\lambda = 1$ and $\mathbf{v} = (0.194, 0.290, 0.387, 0.129)$

b) $\lambda = 9.976$ and $\mathbf{v} = (0.108, 0.158, 0.197, 0.146, 0.178, 0.212)$

Any positive scalar multiple of these eigenvectors can be used (we have chosen state vectors, where the sum of the components are one).

Problem 4.

a) $\lambda^4 - f_1\lambda^3 - p_1 f_2 \lambda^2 - p_1 p_2 f_3 \lambda - p_1 p_2 p_3 f_4 = 0$

b) In the graph of A, we have arrows $1 \to 2 \to 3 \to \cdots \to n$ since $p_1, p_2, \dots, p_{n-1} > 0$, and there is an arrow $n \to 1$ if $f_n > 0$. If $f_n = 0$, there is no path from node n to node 1.

Problem 5.
Dominant eigenvalue $\lambda = 1.562$, see Python runtime example for a Frobenius vector.