

## Key Problems

### Problem 1.

Use Python to find the eigenvalues and eigenvectors of the following matrices. Then check whether the matrices are irreducible, and if so, find the dominant eigenvalue  $\lambda$  and a positive eigenvector  $\mathbf{v}$ :

- a) `A = np.array([[0,0,1/3,1/2],[1/2,0,1/3,1/2],[1/2,1,0,0],[0,0,1/3,0]])`  
 b) `B = np.array([[0,3,0,0,1,2],[3,0,2,2,2,1],[6,4,0,2,1,1],[3,1,1,0,2,2],[2,1,2,4,0,2],[1,2,2,4,4,0]])`

### Problem 2.

Write a Python program that defines a function `Frobenius(matrix)` that has a square non-negative irreducible matrix as input, and a positive eigenvector as output. Check that the function works by using it on the following matrices:

- a) `A = np.array([[0,0,1/3,1/2],[1/2,0,1/3,1/2],[1/2,1,0,0],[0,0,1/3,0]])`  
 b) `B = np.array([[0,3,0,0,1,2],[3,0,2,2,2,1],[6,4,0,2,1,1],[3,1,1,0,2,2],[2,1,2,4,0,2],[1,2,2,4,4,0]])`

### Problem 3.

We consider an  $n \times n$  Leslie matrix  $A$  given by

$$A = \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{pmatrix}$$

where  $0 < p_i < 1$  and  $f_i \geq 0$  for all  $i$ , and  $f_n > 0$ .

- a) Find the characteristic equation of  $A$  in the case  $n = 4$ .  
 b) Explain why the condition  $f_n > 0$  is necessary for  $A$  to be an irreducible non-negative matrix.

### Problem 4.

A population of *rattus norvegicus* is divided into  $n = 21$  age groups (time measured in months), and the growth of the population is given by the Leslie matrix  $A$  with parameters

- ```
f = np.array([0,0,0.3964,1.4939,2.1777,2.5250,2.6282, 2.6749,2.6018,2.4419,2.1865,1.9044,
              1.7259,1.4918,1.2415,0.9522,0.7141,0.4618,0.2518, 0.0901,0.0035])
p = np.array([0.94697,0.99665,0.99926,0.99899,0.99863, 0.99817,0.99753,0.99667,0.99553,
              0.99399,0.99196,0.98926,0.98572,0.98107,0.97511, 0.96748, 0.95797,0.94631,
              0.93247,0.91649])
```

- a) Write a python program that defines a function `Leslie(f,p)` that has one-dimensional arrays  $f$  and  $p$  of size  $n$  and  $n - 1$  as inputs, and the corresponding Leslie matrix  $A$  as output.  
 b) Find the dominant eigenvalue and a Frobenius vector of  $A$ .  
 c) Plot the population growth of all age groups of this population when the initial population has 5 individuals in each age group.

## Answers to Key Problems

### Problem 1.

Both matrices are irreducible, and we find

a)  $\lambda = 1$  and  $\mathbf{v} = (0.194, 0.290, 0.387, 0.129)$

b)  $\lambda = 9.976$  and  $\mathbf{v} = (0.108, 0.158, 0.197, 0.146, 0.178, 0.212)$

Any positive scalar multiple of these eigenvectors can be used (we have chosen state vectors, where the sum of the components are one).

### Problem 3.

a)  $\lambda^4 - f_1\lambda^3 - p_1f_2\lambda^2 - p_1p_2f_3\lambda - p_1p_2p_3f_4 = 0$

b) In the graph of  $A$ , we have arrows  $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n$  since  $p_1, p_2, \dots, p_{n-1} > 0$ , and there is an arrow  $n \rightarrow 1$  if  $f_n > 0$ . If  $f_n = 0$ , there is no path from node  $n$  to node 1.

### Problem 4.

Dominant eigenvalue  $\lambda = 1.562$ , see Python runtime example for a Frobenius vector.