Key Problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ:

a) $A = \begin{pmatrix} 5 & 9 \\ 9 & 5 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -4 \\ 3 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ f) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Problem 2.
Determine whether the matrix A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible:

a) $A = \begin{pmatrix} 5 & 9 \\ 9 & 5 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -4 \\ 3 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ f) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.
Use eigenvalues and eigenvectors of A to determine the limit of A^m when $m \to \infty$, if the limit exists. What can you say about the limit of $A^m \cdot \mathbf{v}_0$, the equilibrium state of the Markov chain with transition matrix A?

a) $A = \begin{pmatrix} 0.40 & 0.15 \\ 0.60 & 0.85 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$ c) $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$

Exercise Problems

Problems from the textbook:	[E] 4.1 - 4.8
Exam problems:	[Midterm 10/2018] Question 1-6

Answers to Key Problems

Problem 1.

a. Eigenvalues $\lambda_1 = -4,\ \lambda_2 = 14$ and eigenvectors $E_{-4} = \text{span}(\mathbf{v}_1)$ and $E_{14} = \text{span}(\mathbf{v}_2)$, where $\mathbf{v}_1 = (-1{,}1)$ and $\mathbf{v}_2 = (1{,}1)$.

b. Eigenvalues $\lambda_1 = \lambda_2 = 3$ and eigenvectors $E_3 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = (1{,}1)$.

c. No eigenvalues or eigenvectors.

d. Eigenvalues $\lambda_1 = \lambda_2 = 4,\ \lambda_3 = 2$ and eigenvectors $E_4 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_2 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = (0{,}1{,}0)$, $\mathbf{v}_2 = (1{,}0{,}1)$, and $\mathbf{v}_3 = (-1{,}0{,}1)$.

e. Eigenvalues $\lambda_1 = \lambda_2 = -1,\ \lambda_3 = 2$ and eigenvectors $E_{-1} = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_2 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = (-1{,}1{,}0)$, $\mathbf{v}_2 = (-1{,}0{,}1)$, and $\mathbf{v}_3 = (1{,}1{,}1)$.

f. Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and eigenvectors $E_0 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = (1{,}0{,}0)$.

Problem 2.

a) Yes, with $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 0 \\ 0 & 14 \end{pmatrix}$

b) No

c) No

d) Yes, with $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

e) Yes, with $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

f) No

Problem 3.

The eigenvalues of A are $\lambda_1 = \lambda_2 = 5$, $\lambda_3 = -1$ and $\lambda_4 = -3$.

Problem 4.

In all cases, $A^m\,\mathbf{v}_0 \to \mathbf{v}$ as $m \to \infty$ for the vector given below, and the limit of A^m is a matrix with the vector $\mathbf{v}$ in each column.

a) $\mathbf{v} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$

b) $\mathbf{v} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

c) $\mathbf{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$