Key Problems

Problem 1.
We consider a subset $\mathcal{L}$ of the vectors $\mathbf{v}_1 = (1,3,4)$, $\mathbf{v}_2 = (-1,3,4)$, $\mathbf{v}_3 = (5,3,4)$, $\mathbf{v}_4 = (6,4,5)$, $\mathbf{v}_5 = (4,2,3)$. In each case, determine whether the vectors in $\mathcal{L}$ are linearly independent, and compute the dimension and find a base of the vector space $V = \text{span}(\mathcal{L})$:

a) $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2\}$
b) $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
c) $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$
d) $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$
e) $\mathcal{L} = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
f) $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

Problem 2.
Find a parametric description of the line through the points $(1,3,2,5)$ and $(-2,4,5,1)$ in $\mathbb{R}^4$. Determine the intersection points (x,y,z,w) of this line and the hyperplane $x + z + w = 0$.

Problem 3.
We consider the 3×5 matrix A given by

$$A = \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 2 & 4 & -2 & -2 & -2 \\ 3 & 5 & -1 & -1 & -1 \end{pmatrix}$$

Compute $\dim V$ and find a base $\mathcal{B}$ of V in each case, and give a geometric characterization of V.

a) $V = \text{Null}(A)$
b) $V = \text{Col}(A)$
c) $V = \text{Row}(A)$

Problem 4.
Let A be a 8×8 matrix with rank $\text{rk}(A) = 7$ and let $\mathbf{b}$ be a vector in $\mathbb{R}^8$. Determine:

a) $\dim \text{Null}(A)$
b) $\dim \text{Col}(A)$
c) $\dim \text{Row}(A)$
d) The number of solutions of $A\mathbf{x} = \mathbf{0}$
e) The number of solutions of $A\mathbf{x} = \mathbf{b}$
f) The number of solutions of $A\mathbf{x} = \mathbf{0}$ that satisfies $x_1 + x_2 + \cdots + x_8 = 1$

Exercise problems

Problems from the textbook:	[E] 2.1 - 2.2, 2.4, 2.5abc, 2.6 - 2.16
Exam problems:	[Midterm 10/2019] Question 1, 2, 8

Answers to Key Problems

Problem 1.

a) Yes, $\dim V = 2$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$

b) No, $\dim V = 2$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$

c) Yes, $\dim V = 3$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

d) Yes, $\dim V = 3$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$

e) Yes, $\dim V = 3$, and $\mathcal{B} = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

f) No, $\dim V = 3$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

Problem 2.

Parametric description: $(x,y,z,w) = (1-3t, 3+t, 2+3t, 5-4t)$. Intersection point: $(x,y,z,w) = (-5,5,8,-3)$.

Problem 3.

a) $\dim \mathrm{Null}(A) = 2$, $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a base for $\mathrm{Null}(A)$ with $\mathbf{w}_1 = (-3,2,1,0,0)$, $\mathbf{w}_2 = (-6,4,0,1,1)$, and $\mathrm{Null}(A)$ is a plane in $\mathbb{R}^5$

b) $\dim \mathrm{Col}(A) = 3$, $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a base of $\mathrm{Col}(A)$ when $\mathbf{v}_i$ are the column vectors of A for $i = 1,2,3,4,5$, and $\mathrm{Col}(A)$ is all of the 3-dimensional space $\mathbb{R}^3$.

c) $\dim \mathrm{Row}(A) = 3$, the three row vectors of A is a base $\mathcal{B}$ of $\mathrm{Row}(A)$, and $\mathrm{Row}(A)$ is a 3-dimensional linear subspace of $\mathbb{R}^5$.

Problem 4.

a) 1

b) 7

c) 7

d) Infinitely many solutions (one degree of freedom)

e) Infinitely many solutions (one degree of freedom) if $\mathbf{b}$ is in $\mathrm{Col}(A)$, otherwise no solutions

f) No solutions if $(1,1,\dots,1)$ is in $\mathrm{Row}(A)$, otherwise one unique solution.