EXAMINATION QUESTION PAPER - Course paper

ELE 37811
Mathematics - Elective

Department of Economics

Start date:	19.10.2020	Time 09.00
Finish date:	26.10.2020	Time 12.00

Weight:	20% of ELE 3781
Total no. of pages:	3 incl. front page
No. of attachments files to question paper:	0
To be answered:	Individually

Answer paper size:	No limit. excl. attachments
Max no. of answer paper attachment files:	1
Allowed answer paper file types:	pdf
Allowed answer paper attachment file types:	txt

Home exam	Midterm exam in ELE 3781 Mathematics elective
Deadline	October 26th, 2020 at 1200

Your answers should be provided as a single file in pdf format, and a single attachment in txt format. The attachment should contain python code, and the main file should contain the rest of your answer. It is almost always best to write mathematics by hand and scan to pdf.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

The python code below defines a function `f`. Explain what this function does when called with a 2-dimensional NumPy array as argument.

```
def f(matrix):
  if matrix.shape[0] != matrix.shape[1]:
    print("Must be a square matrix")
    return(-1)
  if matrix.shape[0] == 1:
    return matrix[0,0]
  else:
    return matrix[0,0] + f(matrix[1:,1:])
```

Question 2.

In this question, you may use Table A.1 in Eriksen [E] to find values of trigonometric functions.

a) Let ω be the complex number with polar coordinates given by $r = 1$ and $\theta = 120°$. Write $z = \omega$ and $z = \omega^2$ in the form $z = a + ib$, and draw a figure of the complex plane where the points $z = 1$, $z = \omega$ and $z = \omega^2$ are marked.

b) Explain that $1, \omega, \omega^2$ are the complex solutions of $x^3 = 1$. Use this to show that if $z = z^*$ is one solution of the equation $x^3 = -i$, then the other solutions of this equation are $z = z^* \cdot \omega$ and $z = z^* \cdot \omega^2$.

c) Compute $(2 + i)^3$, and use this computation to find all complex solutions of $x^3 = 2 + 11i$.

d) When Cardano's formula is applied to the equation $x^3 = 15x+4$, it gives the following expression for solutions:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Some background on Cardano's formula is given in the text below. Find a way to interpret the expression above so that it makes sense, and use this to find all complex solutions of $x^3 = 15x+4$ expressed in the form $z = a + ib$.

Cardano's formula

Let $x^3 = px + q$ be a cubic equation with real coefficients p, q. Cardano's formula for solutions is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

In fact, if we substitute $x = u + v$ in the equation, we get $u^3 + 3u^2v + 3uv^2 + v^3 = p(u + v) + q$, and this can be written as

$$(u^3 + v^3 - q) + (u + v)(3uv - p) = 0$$

Hence choosing u and v such that $u^3 + v^3 - q = 0$ and $3uv - p = 0$ will be sufficient for $x = u + v$ to be a solution. The second equation gives $v = p/(3u)$, and we substitute this in the first equation:

$$u^3 + \left(\frac{p}{3u}\right)^3 - q = 0 \quad \Rightarrow \quad u^6 - qu^3 + \left(\frac{p}{3}\right)^3 = 0$$

This is a quadratic equation in u^3, and we find that

$$u^3 = \frac{q \pm \sqrt{(-q)^2 - 4(p/3)^3}}{2} = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$$

Since $u^3 + v^3 = q$, we may without loss of generality assume that

$$u^3 = \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}, \quad v^3 = \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$$

Since $x = u + v$, we add the third roots of the expressions above, and this gives Cardano's formula:

$$x = u + v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Question 3.
Write python code that defines two functions `rank` and `pivots` following the specifications below:

a) The function `rank(matrix)` returns the rank of a 2-dimensional NumPy array `matrix`.
b) The function `pivots(matrix)` returns the pivots of a 2-dimensional NumPy array `matrix` as a list of all pivot positions (i, j).

Executable python code that define both functions should be included as an attachment. Use these functions to compute the rank and pivots of the matrices A and B corresponding to the NumPy arrays given below.

```
A = np.array([[1,1,1,3,-1],[1,2,4,7,3],[2,3,5,10,2]])
B = np.array([[1,3,1],[1,4,3],[2,3,5],[-1,10,2]])
```