EVALUATION GUIDELINES - Course paper

ELE 37811
Mathematics - Elective

Department of Economics

Start date:	19.10.2020	Time 09:00
Finish date:	26.10.2020	Time 12:00

For more information about formalities, see examination paper.

Solutions	Midterm exam in ELE 3781 Mathematics elective
Deadline	October 26th, 2020 at 1200

Question 1.

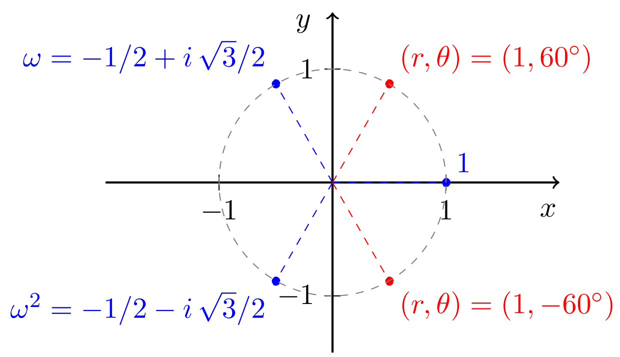
The python code will give an error message if the argument is not a square matrix. If the matrix A is square, then the function `f` will return the trace $\text{tr}(A)$ of A when called with the 2-dimensional NumPy array corresponding to A: If $A = (a_{11})$ is a 1×1 matrix, then it will return $\text{tr}(A) = a_{11}$. If A is an $n \times n$ matrix with $n > 1$, it will use the recursive formula

$$\text{tr}(A) = \text{tr}\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = a_{11} + \text{tr}\begin{pmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

to compute and return the trace $\text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$ of A. This will be done by calling `f` recursively.

Question 2.

Using Table A.1 in Eriksen [E] and symmetry, we find that $\sin(120°) = \sin(60°) = \sqrt{3}/2$ and $\cos(120°) = -\cos(60°) = -1/2$. In fact, the points with polar coordinates $(1, 120°)$ and $(1, 60°)$ are reflections of each other along the y-axis, and this reflection maps (x, y) to $(-x, y)$; see the figure below where the points are marked in blue and red.



a) We have that $\omega = 1 \cdot (\cos 120° + i \sin(120°) = -1/2 + i\sqrt{3}/2$. To express ω^2 in this form, we could use the polar coordinates $(1, 240°)$ of ω^2, and $\cos(240°) = \cos(-120°) = \cos(120°) = -1/2$ and $\sin(240°) = \sin(-120°) = -\sin(120°) = -\sqrt{3}/2$ to obtain

$$\omega^2 = 1 \cdot (\cos 240° + i \sin(240°) = -1/2 - i\sqrt{3}/2$$

Alternatively, we could use multiplication to find ω^2:

$$\omega^2 = (-1/2 + i\sqrt{3}/2)^2 = 1/4 - 2i\sqrt{3}/4 + 3i^2/4 = -1/2 - i\sqrt{3}/2$$

The complex numbers 1, ω and ω^2 are shown on the figure above.

b) To solve the equation $x^3 = 1$, we write $1 = e^{i \cdot 0°}$ since 1 has polar coordinates $(1, 0°)$. Let (r, θ) be the polar coordinates of x, then $x^3 = r^3 e^{3\theta}$, and $x^3 = 1$ can be written

$$r^3 e^{3\theta} = 1 \cdot e^{i \cdot 0°} \quad \Rightarrow \quad r^3 = 1 \text{ and } 3\theta = 0° + k \cdot 360°$$

This gives $r = 1$ and $\theta = k \cdot 120°$, and $k = 0, 1, 2$ give the three distinct solutions of $x^3 = 1$. Hence the solutions are $x = 1$, $x = \omega$, and $x = \omega^2$ (for $k = 0, 1, 2$). For the second part, assume that z^* is one solution of the equation $x^3 = -i$. Then we have that

$$(z^*\omega)^3 = (z^*)^3\omega^3 = -i \cdot 1 = -i, \quad (z^*\omega^2)^3 = (z^*)^3\omega^6 = -i \cdot 1 = -i$$

Since the third order equation $x^3 = -i$ has three complex roots, the roots must be z^*, $z^*\omega$, and $z^*\omega^2$.

c) We have that $(2+i)^2 = (4+4i-1) = 3+4i$, and therefore that

$$(2+i)^3 = (2+i)^2(2+i) = (3+4i)(2+i) = 6+8i+3i+4i^2 = 2+11i$$

This means that $2+i$ is one solution of $x^3 = 2+11i$, and by the same argument as in b) it follows that the three complex solutions of $x^3 = 2+11i$ are given by $(2+i)$, $(2+i)\omega$, and $(2+i)\omega^2$. We can express $(2+i)\omega$ and $(2+i)\omega^2$ as

$$(2+i)(-1/2+i\sqrt{3}/2) = -1-i/2+i\sqrt{3}-\sqrt{3}/2 = (-1-\sqrt{3}/2)+i(\sqrt{3}-1/2)$$
$$(2+i)(-1/2-i\sqrt{3}/2) = -1-i/2-i\sqrt{3}+\sqrt{3}/2 = (-1+\sqrt{3}/2)+i(-\sqrt{3}-1/2)$$

d) Recall that when a is a real number, we have defined $\sqrt[3]{a}$ to be the unique real number x such that $x^3 = a$. There is no definition of this kind for the third root of a complex number $a+ib$ with $b \neq 0$. We must therefore interpret $\sqrt[3]{2+\sqrt{-121}} = \sqrt[3]{2+11i}$ as any complex solution of $x^3 = 2+11i$, and interpret $\sqrt[3]{2-\sqrt{-121}} = \sqrt[3]{2-11i}$ as any complex solution of $x^3 = 2-11i$. According to Cardano's formula, the solutions of $x^3 = 15x+4$ can be written $x = u+v$, where u is a solution of the equation $u^3 = 2+11i$ and v is a solution of $v^3 = 2-11i$. We solved the first equation in c). Using the results we obtained there, we see that the possible values for u are

$$u_1 = 2+i, \quad u_2 = (-1-\sqrt{3}/2)+i(\sqrt{3}-1/2), \quad u_3 = (-1+\sqrt{3}/2)+i(-\sqrt{3}-1/2)$$

In a similar way, solve $v^3 = 2-11i$ to find possible values of v. To compute these solutions, notice that $2-11i = \overline{2+11i}$ is the complex conjugate of $2+11i$. This means that if u is a solution of $u^3 = 2+11i$, then $v = \overline{u}$ gives

$$v^3 = \overline{u}^3 = \overline{u^3} = \overline{2+11i} = 2-11i$$

Hence the possible values for v are the complex conjugates of u_1, u_2, and u_3, given by

$$v_1 = \overline{u_1} = 2-i, \quad v_2 = \overline{u_2} = (-1-\sqrt{3}/2)+i(-\sqrt{3}+1/2), \quad v_3 = \overline{u_3} = (-1+\sqrt{3}/2)+i(\sqrt{3}+1/2)$$

We could also have computed the solutions by multiplying $v_1 = 2-i$ with $1, \omega, \omega^2$, and we see that $v_2 = \omega^2 v_1$ and $v_3 = \omega v_1$. Since $x = u+v$, we can combine the different possible values for u and v. But notice that according to the argument behind Cardano's formula, we must have $u \cdot v = p/3 = 15/3 = 5$. This means that we get the solutions

$$x_1 = u_1 + v_1 = 4$$
$$x_2 = u_2 + v_2 = -2-\sqrt{3}$$
$$x_3 = u_3 + v_3 = -2+\sqrt{3}$$

since $u_1 \cdot v_1 = (2+i)(2-i) = 4+1 = 5$, and therefore $u_2 v_2 = u_1\omega \cdot v_1\omega^2 = 5\,\omega^3 = 5$ and $u_3 v_3 = u_1\omega^2 \cdot v_1\omega = 5\,\omega^3 = 5$. The three complex solutions of $x^3 = 15x+4$ are therefore

$$x_1 = 4, \quad x_2 = -2-\sqrt{3}, \quad x_3 = -2+\sqrt{3}$$

Question 3.

See the next page for the python code for the functions `rank` and `pivots`. We get the following results when using these function on the matrices A and B:

a) $\text{rk}(A) = 2$ and $\text{rk}(B) = 3$

b) The pivot positions of A are $(1,1)$, $(2,2)$. The pivot positions of B are $(1,1)$, $(2,2)$, $(3,3)$.

Midterm-2020-10

October 22, 2020

[1]:
```
# Python code: Gaussian elimination

import numpy as np
```

[2]:
```
# Elementary row operations

def Rswitch(matrix,i,j):
    r = matrix[i-1].copy()
    matrix[i-1] = matrix[j-1]
    matrix[j-1] = r
    return(matrix)

def Rmult(matrix,i,c):
    matrix[i-1]=matrix[i-1]*c
    return(matrix)

def Radd(matrix,i,j,c):
    matrix[j-1]=matrix[j-1] + c*matrix[i-1]
    return(matrix)
```

[3]:
```
# Rank

def rank(matrix):
    # check the number of rows
    if matrix.shape[0]==0:
        return(0)
    if matrix.shape[1]==0:
        return(0)
    # get the leftmost column, nonzero positions
    lcol = matrix[:,0]
    nz = np.arange(lcol.size)[lcol != 0]
    # when zero column, move to next column, if any
    if nz.size==0:
        return(rank(matrix[:,1:]))
    # find first non-zero entry in column
    p=nz[0]
    if p!=0:
```

```
        Rswitch(matrix,1,p+1)
    # get zeros under the pivot
    for r in range(1,lcol.size):
        Radd(matrix,1,r+1,-matrix[r,0]/matrix[0,0])
    return(1+rank(matrix[1:,1:]))
```

[4]:
```
# Pivots

def pivots(matrix):
    r = rank(matrix)
    pivots = []
    for i in range(r):
        row = matrix[i]
        c = np.arange(row.size)[row != 0][0]
        pivots.append((i+1,c+1))
    return(pivots)
```

[5]:
```
# Some tests that you can run

A = np.array([[1,1,1,3,-1],[1,2,4,7,3],[2,3,5,10,2]])
B = np.array([[1,3,1],[1,4,3],[2,3,5],[-1,10,2]])
```

[6]:
```
rank(A)
```

[6]: 2

[7]:
```
rank(B)
```

[7]: 3

[8]:
```
pivots(A)
```

[8]: [(1, 1), (2, 2)]

[9]:
```
pivots(B)
```

[9]: [(1, 1), (2, 2), (3, 3)]

[ ]:
```
```