

# ELE 07811

## Mathematics - Elective

Department of Economics

<b>Start date:</b>	18.01.2021	Time 09:00
<b>Finish date:</b>	18.01.2021	Time 12:00

For more information about formalities, see examination paper.

**Question 1.**

- (a) The nullspace of  $A$  is the set of solutions of  $A\mathbf{x} = \mathbf{0}$ . We use Gaussian elimination to find the solutions:

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 22 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivot positions, we have that  $\dim \text{Null}(A) = 4 - 3 = 1$ , with  $y$  free since there is no pivot position in the  $y$  column. To find solutions  $(x, y, z, w)$  we use back substitution. This gives  $16w = 0$ , or  $w = 0$  from the third equation,  $2z - w = 0$ , or  $z = 0$  from the second, and  $x + 2y + 2w = 0$ , or  $x = -2y$  with  $y$  free from the first. The solutions are therefore given by

$$(x, y, z, w) = (-2y, y, 0, 0) = y \cdot (-2, 1, 0, 0) = y \cdot \mathbf{w}$$

where  $\mathbf{w} = (-2, 1, 0, 0)$ , and  $\mathcal{B} = \{\mathbf{w}\}$  is a base of  $\text{Null}(A)$ . To find the eigenvalues of  $A$ , we solve the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 0 & 2 \\ 2 & 4 - \lambda & 2 & 0 \\ 0 & 0 & 5 - \lambda & 6 \\ 0 & 0 & 6 & 10 - \lambda \end{vmatrix} = 0$$

We use cofactor expansion along the first column to compute the determinant, and get

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda) \cdot (4 - \lambda) \begin{vmatrix} 5 - \lambda & 6 \\ 6 & 10 - \lambda \end{vmatrix} - 2 \cdot 2 \begin{vmatrix} 5 - \lambda & 6 \\ 6 & 10 - \lambda \end{vmatrix} \\ &= [(1 - \lambda)(4 - \lambda) - 4] \cdot [(5 - \lambda)(10 - \lambda) - 36] \\ &= (\lambda^2 - 5\lambda)(\lambda^2 - 15\lambda + 14) = \lambda(\lambda - 5)(\lambda - 1)(\lambda - 14) \end{aligned}$$

Therefore the eigenvalues of  $A$  are  $\lambda_1 = 0$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = 5$ ,  $\lambda_4 = 14$ .

- (b) Let  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  be the column vectors of  $A$ . We see from the echelon form above that there are pivot positions in column 1, 3 and 4 of  $A$ . Hence  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$  is a base of  $\text{Col}(A)$ . Since  $\dim \text{Col}(A) = 3 < 4 = \dim \mathbb{R}^4$ , there are vectors in  $\mathbb{R}^4$  not in  $\text{Col}(A)$ . We use Gaussian elimination to find all vectors  $(a, b, c, d)$  that are in  $\text{Col}(A)$ :

$$\left( \begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 2 & 4 & 2 & 0 & b \\ 0 & 0 & 5 & 6 & c \\ 0 & 0 & 6 & 10 & d \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 0 & 0 & 2 & -4 & b - 2a \\ 0 & 0 & 10 & 12 & 2c \\ 0 & 0 & 6 & 10 & d \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 0 & 0 & 2 & -4 & b - 2a \\ 0 & 0 & 0 & 32 & 2c - 5b + 10a \\ 0 & 0 & 0 & 22 & d - 3b + 6a \end{array} \right)$$

This linear system has solutions if and only if  $32(d - 3b + 6a) = 22(2c - 5b + 10a)$ , which can be written  $-28a + 14b - 44c + 32d = 0$ , since a non-zero determinant correspond to a pivot position in the fourth and fifth column. We can write this equation as  $-14a + 7b - 22c + 16d = 0$ , and this means that  $\mathbf{v} = (a, b, c, d)$  is not in  $\text{Col}(A)$  if and only if  $-14a + 7b - 22c + 16d \neq 0$ . One example of a vector not in  $\text{Col}(A)$  is  $\mathbf{v} = (1, 0, 0, 0)$  since  $-14 \cdot 1 = -14 \neq 0$ .

- (c) The symmetric matrix  $B$  of the quadratic form  $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  is given by

$$B = \frac{1}{2}(A + A^T) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \\ 0 & 1 & 5 & 6 \\ 1 & 0 & 6 & 10 \end{pmatrix}$$

and the first three leading principal minors of  $B$  are  $D_1 = 1$ ,  $D_2 = 1 \cdot 4 - 2 \cdot 2 = 0$ , and

$$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 5 \end{vmatrix} = 1(20 - 1) - 2(10 - 0) = 19 - 20 = -1$$

Since  $D_1 > 0$  and  $D_3 < 0$ , the quadratic form  $f$  is **indefinite**.

- (d) Let us write  $\mathbf{x} = (x, y, z, w)$  for the variables. Then the constraint is  $x^2 + y^2 + z^2 + w^2 = 1$  since  $\mathbf{x}^T \mathbf{x} = \mathbf{x}^T I \mathbf{x}$ . Hence the set of admissible points is closed and bounded, and the Lagrange problem has a solution. It is also clear that there are no admissible points where NDCQ fails, since

$$\text{rk } J = \text{rk} \begin{pmatrix} 2x & 2y & 2z & 2w \end{pmatrix} = \begin{cases} 0, & (x, y, z, w) = (0, 0, 0, 0) \\ 1, & (x, y, z, w) \neq (0, 0, 0, 0) \end{cases}$$

and  $(0, 0, 0, 0)$  is clearly not admissible since  $\mathbf{x}^T \mathbf{x} = 0 \neq 1$  at this point. Hence the problem has a max and a min among the ordinary candidate points  $(\mathbf{x}; \lambda)$  where the first order conditions hold. The Lagrangian is  $\mathcal{L} = f(\mathbf{x}) - \lambda(\mathbf{x}^T \mathbf{x})$ , and the first order conditions can be written in matrix form as

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = 2B\mathbf{x} - \lambda \cdot 2I\mathbf{x} = 2(B - \lambda I)\mathbf{x} = \mathbf{0}$$

where  $B$  is the symmetric matrix of the quadratic form  $f$ . We could also find the first order conditions by writing out the quadratic form  $f$  and the constraint. This implies that  $\mathbf{x} = \mathbf{0}$  or that  $\mathbf{x}$  is an eigenvector of  $B$  with eigenvalue  $\lambda$ . Clearly,  $\mathbf{x} = \mathbf{0}$  does not satisfy the constraint. In fact, the constraint  $\mathbf{x}^T \mathbf{x} = 1$  means that  $\mathbf{x}$  is a vector of length one. We notice that the matrix  $B$  has four eigenvalues counted with multiplicity since it is symmetric, and we can write them  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ . For each eigenvalue  $\lambda$ , there are at least two eigenvectors of length one with eigenvalue  $\lambda$ . In fact, there are two if  $\dim E_\lambda = 1$  and infinitely many otherwise. Based on this, we conclude that there are at least two candidate points  $(\mathbf{x}; \lambda)$  for each eigenvalue  $\lambda$  where  $\mathbf{x}$  is an eigenvector of  $B$  with eigenvalue  $\lambda$ , and that these are the only candidate points. The value of  $f$  at such a candidate point is given by

$$f(\mathbf{x}) = \mathbf{x}^T B \mathbf{x} = \mathbf{x}^T \lambda \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} = \lambda \cdot 1 = \lambda$$

Hence the maximum value in the Lagrange problem is given by  $f_{\max} = \lambda_4$  (the maximal eigenvalue of  $B$ ), and the minimum value is given by  $f_{\min} = \lambda_1$  (the minimal eigenvalue of  $B$ ).

### Question 2.

- (a) We have that  $f(x, y, z) = \ln(5 - u)$  where  $u = u(x, y, z) = x^2 + y^2 + z^2 - xy + xz - yz$  is a quadratic form. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}$$

Since the leading principal minors are  $D_1 = 1$ ,  $D_2 = 1 - 1/4 = 3/4$ , and

$$D_3 = |A| = 1(1 - 1/4) + 1/2(-1/2 + 1/4) + 1/2(1/4 - 1/2) = 3/4 - 1/8 - 1/8 = 1/2$$

with  $D_1, D_2, D_3 > 0$ ,  $u$  is positive definite. This means that  $u(x, y, z) \geq 0$ , and therefore that  $5 - u(x, y, z) \leq 5$  at all points  $(x, y, z)$ . We conclude that  $f_{\max} = \ln 5$  since  $h(u) = \ln(u)$  is a strictly increasing function; that is,  $u_1 < u_2$  implies that  $h(u_1) < h(u_2)$  since  $h'(u) = 1/u > 0$ . The maximizer is  $(x, y, z) = (0, 0, 0)$  with  $f(0, 0, 0) = \ln 5$ .

- (b) A subset of  $D \subseteq \mathbb{R}^4$  is compact if and only if it is closed and bounded, and by definition, it is bounded if there are finite numbers  $a_1, a_2, a_3, a_4$  and  $b_1, b_2, b_3, b_4$  such that

$$a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3, a_4 \leq w \leq b_4$$

for all  $(x, y, z, w) \in D$ . Since the point  $(a, a, -a, -a)$  give  $xw + yz = -2a^2$ , and  $-2a^2 \leq -2$  when  $a^2 \geq 1$ , we have that  $(a, a, -a, -a) \in D$  for all  $a \geq 1$  and all  $a \leq -1$ . This means that  $D$  is not bounded, and therefore **not compact**. The set  $D$  is closed since it is given by a closed inequality.

- (c) The Kuhn-Tucker problem can be written in standard form as

$$\max -f(x, y, z, w) = -x^2 - 4y^2 - 9z^2 - w^2 \text{ subject to } xw + yz \leq -2$$

Its Lagrangian is  $\mathcal{L} = -x^2 - 4y^2 - 9z^2 - w^2 - \lambda(xw + yz)$ , and the first order conditions (FOC) are

$$\begin{aligned}\mathcal{L}'_x &= -2x - \lambda w = 0 & \mathcal{L}'_y &= -8y - \lambda z = 0 \\ \mathcal{L}'_z &= -18z - \lambda y = 0 & \mathcal{L}'_w &= -2w - \lambda x = 0\end{aligned}$$

The constraint (C) is given by  $xw + yz \leq -2$ , and the complementary slackness conditions (CSC) by  $\lambda \geq 0$  and  $\lambda(xw + yz + 2) = 0$ . The first and last FOC's give

$$x = -\frac{1}{2}\lambda w \quad \Rightarrow \quad -2w = \lambda \left( -\frac{1}{2}\lambda w \right) = -\frac{1}{2}\lambda^2 w$$

This means that  $w = \lambda^2 w/4$ , which give  $w = 0$  or  $\lambda^2 = 4$ . Since  $\lambda \geq 0$  by the CSC, this implies that either  $w = x = 0$ , or  $\lambda = 2$  and  $x = -w$ . In a similar way, the two middle FOC's give

$$y = -\frac{1}{8}\lambda z \quad \Rightarrow \quad -18z = \lambda \left( -\frac{1}{8}\lambda z \right) = -\frac{1}{8}\lambda^2 z$$

This means that  $z = \lambda^2 z/144$ , which give  $z = 0$  or  $\lambda^2 = 144$ . Since  $\lambda \geq 0$  by the CSC, this implies that either  $z = y = 0$ , or  $\lambda = 12$  and  $y = -3z/2$ . From the constraint, we see that the point  $(x, y, z, w) = (0, 0, 0, 0)$  is not admissible, and we are left with the following possibilities:

$$(x, y, z, w; \lambda) = (0, -3z/2, z, 0; 12), (-w, 0, 0, w; 2)$$

By the second part of the CSC, we have that  $xw - yz = -2$  since  $\lambda > 0$ . In the first case, we get that  $xw + yz = -3z/2 \cdot z = -3z^2/2 = -2$ , or  $z^2 = 4/3$ . In the second case, we get that  $xw + yz = -w \cdot w = -w^2 = -2$ , or  $w^2 = 2$ . Hence the following points are the solutions of the Kuhn-Tucker conditions, with corresponding  $f$ -values:

$$\begin{aligned}(x, y, z, w; \lambda) &= (0, -\sqrt{3}, 2/\sqrt{3}, 0; 12), (0, \sqrt{3}, -2/\sqrt{3}, 0; 12), & f &= 4 \cdot 3 + 9 \cdot 4/3 = 24 \\ &= (-\sqrt{2}, 0, 0, \sqrt{2}; 2), (\sqrt{2}, 0, 0, -\sqrt{2}; 2), & f &= 2 + 2 = 4\end{aligned}$$

### Question 3.

- (a) We solve the second order linear difference equation  $y_{t+2} - y_{t+1} - 2y_t = 4t$  using superposition. To find  $y_t^h$ , we consider the homogeneous equation  $y_{t+2} - y_{t+1} - 2y_t = 0$ , which has characteristic equation  $r^2 - r - 2 = 0$  with roots  $r = -1$  and  $r = 2$ . Hence  $y_t^h = C_1(-1)^t + C_2 \cdot 2^t$ . To find the particular solution  $y_t^p$ , we consider the difference equation  $y_{t+2} - y_{t+1} - 2y_t = 4t$  and use the method of undetermined coefficients. Since  $f_t = 4t$ , we use  $y_t = At + B$ , which gives  $y_{t+1} = A(t+1) + B$  and  $y_{t+2} = A(t+2) + B$ . When we substitute this into the difference equation, we get

$$A(t+2) + B - (A(t+1) + B) - 2(At + B) = 4t \quad \Leftrightarrow \quad -2At + (A - 2B) = 4t$$

Comparing coefficients, we get  $-2A = 4$ , or  $A = -2$ , and  $A - 2B = 0$ , or  $B = A/2 = -1$ . Hence  $y_p = -2t - 1$  and the general solution is therefore

$$y = y_t^h + y_t^p = C_1(-1)^t + C_2 \cdot 2^t - 2t - 1$$

With initial conditions  $y_0 = y_1 = 1$ , we get  $C_1 + C_2 - 1 = 1$ , or  $C_1 + C_2 = 2$  from the first condition, and  $-C_1 + 2C_2 - 3 = 1$ , or  $-C_1 + 2C_2 = 4$  from the second. Adding these equations, we get  $3C_2 = 6$ , or  $C_2 = 2$ , and this gives  $C_1 = 0$ . The particular solution is therefore

$$y = 2 \cdot 2^t - 2t - 1 = 2^{t+1} - 2t - 1$$

This means that  $y_{17} = 2^{18} - 2(17) - 1 = 262\,109$ .

- (b) The differential equation  $t^2 y' + 2ty = 1$  can be written  $t^2 y' = 1 - 2ty$  or  $y' = (1 - 2ty)/t^2$ . The right-hand side cannot be factored as  $f(t) \cdot g(y)$ , hence the equation is **not separable**. The differential equation can be written on the form  $y' + a(t)y = b(t)$ , since division by  $t^2$  gives

$$y' + \frac{2}{t}y = \frac{1}{t^2}$$

Hence the equation is **linear**. We solve it using integrating factor: The integrating factor is  $u = e^{2 \ln t} = (e^{\ln t})^2 = t^2$  since  $\int 2/t dt = 2 \ln |t| + C$  and  $|t|^2 = t^2$ . Multiplying with  $u = t^2$ , we get

$$t^2 y' + 2ty = 1 \Leftrightarrow (t^2 y)' = 1 \Leftrightarrow t^2 y = \int 1 dt = t + C$$

The equation has general solution  $y = 1/t + C/t^2$ . Finally, the differential equation can be written  $t^2 y' + 2ty - 1 = 0$ , and with  $h = t^2 y - t$  we have

$$t^2 y' + (2ty - 1) = h'_y \cdot y' + h'_t = 0$$

Therefore the equation is **exact**, and the solution is given by  $h(t, y) = C$ . This gives

$$t^2 y - t = C \Leftrightarrow y = \frac{C + t}{t^2} = C/t^2 + 1/t$$

We notice that this is the same solution as we found above using integrating factor.

- (c) We can write  $y = 3e^{-2t} - 5e^t + 12e^{-3t} = y_h + y_p$  in several different ways. If the general homogeneous solution is  $y_h = C_1 e^{-2t} - C_2 e^t$ , then the characteristic roots are  $r = -2$  and  $r = -1$ , which gives the characteristic equation  $(r + 2)(r - 1) = r^2 + r - 2 = 0$ . Hence the homogeneous equation is  $y'' + y' - 2y = 0$ . If we substitute  $y = y_p = 12e^{-3t}$  in the left-hand side, we get

$$y'' + y' - 2y = (-3)^2 \cdot 12e^{-3t} + (-3) \cdot 12e^{-3t} - 2 \cdot 12e^{-3t} = 12e^{-3t}(9 - 3 - 2) = 48e^{-3t}$$

It follows that the linear second order differential equation  $y'' + y' - 2y = 48e^{-3t}$  has  $y$  as a particular solution. Using the other decompositions  $y = y_h + y_p$ , we find that the linear second order differential equations

$$y'' + 5y' + 6y = -60e^t \quad \text{or} \quad y'' + 2y' - 3y = -9e^{-2t}$$

also have  $y$  as a particular solution.

- (d) We consider the system  $\mathbf{y}' = A\mathbf{y}$  with  $\mathbf{y} = (y, y', y'')$ . It has the form

$$\mathbf{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix} \cdot \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = A\mathbf{y}$$

where the coefficients in the first and second row follows from the fact that  $y'_1 = y_2$  and  $y'_2 = y_3$  when  $(y_1, y_2, y_3) = (y, y', y'')$ . We know that when  $A$  is diagonalizable with eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ , then the general solution of the system is given by

$$\mathbf{y} = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + C_3 \mathbf{v}_3 e^{\lambda_3 t}$$

Comparing with  $y = 3e^{-2t} - 5e^t + 12e^{-3t}$  from (c), we see that we should try to find a diagonalizable matrix  $A$  of the above form with eigenvalues  $\lambda_1 = -2$ ,  $\lambda_2 = 1$ , and  $\lambda_3 = -3$ . We use this to determine  $r, s, t$  in the last row of  $A$ . In fact, the eigenvalues of  $A$  are given by the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ r & s & t - \lambda \end{vmatrix} = 0$$

Cofactor expansion along the first row gives

$$0 = -\lambda(-\lambda(t - \lambda) - s) - 1(0 - r) = -\lambda(\lambda^2 - t\lambda - s) + r = -\lambda^3 + t\lambda^2 + s\lambda + r$$

On the other hand, since the eigenvalues of  $A$  should be  $\lambda_1 = -2$ ,  $\lambda_2 = 1$ , and  $\lambda_3 = -3$ , the characteristic equation is given by

$$-(\lambda + 2)(\lambda - 1)(\lambda + 3) = -(\lambda^2 + \lambda - 2)(\lambda + 3) = -(\lambda^3 + 4\lambda^2 + \lambda - 6) = 0$$

This can be written  $-\lambda^3 - 4\lambda^2 - \lambda + 6 = 0$ . Comparing the two characteristic equations, we find that  $t = -4$ ,  $s = -1$ , and  $r = 6$ . This means that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}$$

We see that this matrix is diagonalizable with three distinct eigenvalues.