Solutions Challenging Matrix Problems

(Problem Sheet 4, Problem 13–15)

13.

$$\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$$

$$x(x^2-9) - 2(2x-6) + 3(6-2x) = 0$$
$$x^3 - 9x - 4x + 12 + 18 - 6x = 0$$
$$x^3 - 19x + 30 = 0$$

We see that $x=2$ and $x=3$ are solutions (since $x=2$ gives row 1 = row 2 and $x=3$ gives row 2 = row 3). Therefore

$$x^3 - 19x + 30 = (x-2)(x-3)\cdot Q(x)$$

where $Q(x) = (x+5)$. This follows by polynomial division or by the fact that $(-2)\cdot(-3)\cdot 5 = 30$. Hence

$$x^3 - 19x + 30 = 0$$
$$(x-2)(x-3)(x+5) = 0$$
$$\underline{\underline{x=2,\ x=3,\ x=-5}}$$

14.

$$\begin{vmatrix} x+1 & 0 & x & 0 & x-1 & 0 \\ 0 & x & 0 & x-1 & 0 & x+1 \\ x & 0 & x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 & 0 & x \\ x-1 & 0 & x+1 & 0 & x & 0 \\ 0 & x+1 & 0 & x & 0 & x-1 \end{vmatrix} \begin{matrix} -1 \\ -1 \end{matrix} = 9$$

$$\begin{matrix} -x+1 & -x \end{matrix} \begin{vmatrix} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ x & 0 & x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 & 0 & x \\ x-1 & 0 & x+1 & 0 & x & 0 \\ 0 & x+1 & 0 & x & 0 & x-1 \end{vmatrix} \begin{matrix} -x+1 \\ -x-1 \end{matrix} = 9$$

$$2 \begin{vmatrix} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 3x+1 & 0 \\ 0 & 0 & 0 & 3x-1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3x-2 & 0 \\ 0 & 0 & 0 & 3x+2 & 0 & -2 \end{vmatrix} = 9$$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 3x+1 & 0 \\ \hline 0 & 0 & 0 & 3x-1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 9x & 0 \\ 0 & 0 & 0 & 3x+2 & 0 & -2 \end{array}\right| = 9$$

$$1 \cdot 1 \cdot (-1) \cdot 9x \cdot \left(-2(3x-1)-1(3x+2)\right) = 9$$

$$-9x(-9x) = 9$$

$$x^2 = 1/9$$

$$\underline{\underline{x = \pm 1/3}}$$

15. Equation: Equation no:

Equation	Equation no:
$x_2 + x_3 + \dots + x_{n-1} + x_n = 2$	(1)
$x_1 + x_3 + \dots + x_n = 4$	(2)
$x_1 + x_2 + \dots + x_n = 6$	(3)
⋮	⋮
$x_1 + x_2 + x_3 + \dots + x_{n-1} = 2n$	(n)

We add all n equations, and get

$$(n-1)x_1 + (n-1)x_2 + \dots + (n-1)x_n = 2+4+6+\dots+2n$$

$$(n-1)\cdot(x_1+x_2+\dots+x_n) = n \cdot \frac{2+2n}{2} = n\cdot(n+1)$$

$$x_1+x_2+\dots+x_n = \frac{n\cdot(n+1)}{(n-1)}$$

If we subtract $x_1+x_2+\dots+x_n = \frac{n(n+1)}{n-1}$ from equation (i) we get:

$$-x_i = 2i - \frac{n(n+1)}{n-1}$$

$$\underline{\underline{x_i = \frac{n(n+1)}{n-1} - 2i}}$$

$$\left(\underline{x_1 = \frac{n(n+1)}{n-1} - 2},\ \underline{x_2 = \frac{n(n+1)}{n-1} - 4},\ \underline{x_3 = \frac{n(n+1)}{n-1} - 6},\ \dots,\ \underline{x_n = \frac{n(n+1)}{n-1} - n} \right)$$