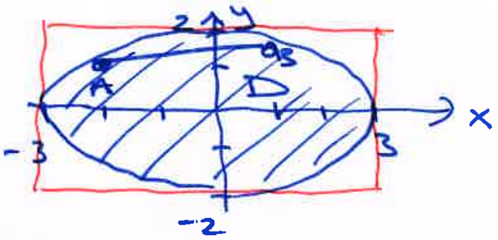
Plan	
1 Key Problems:	7.1bd, 8.1ab, 9.1, 9.2b, 9.3, 9.4
2 Final Exams:	11/2019 Q4 ; 06/2018 Q4

① Key Problems

7.1 b) $D = \{(x,y) : 4x^2 + 9y^2 \leq 36\}$



$4x^2 + 9y^2 = 36 \quad | : 36$

$\frac{4x^2}{36} + \frac{9y^2}{36} = 1$ $\qquad \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ ellipse, center $(0,0)$ half-axes $a=3$, $b=2$

D bounded since

$-3 \leq x \leq 3$

$-2 \leq y \leq 2$

D is convex since

$A, B \in D \Rightarrow [A, B] \subseteq D$

Without the figure:

$4x^2 + 9y^2 \leq 36$

$4x^2 \leq 36 \quad x^2 \leq 9$

$9y^2 \leq 36 \quad y^2 \leq 4$

$\boxed{\begin{array}{l} -3 \leq x \leq 3 \\ -2 \leq y \leq 2 \end{array}}$

D is bounded

Helpful to know: 2 var's

straight line $\Leftrightarrow$ linear eqn.

circle / ellipse $\Leftrightarrow$ $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

hyperbola / parabola : $y = 1/x$, $y = x^2$

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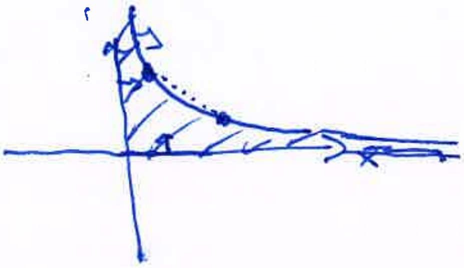
d) $D = \{(x,y) : 4xy \leq 1,\ \underline{x, y > 0}\}$



$4xy = 1 \quad y = \frac{1}{4x} = \frac{1}{4}\cdot\frac{1}{x}$

D is <u>not bounded</u> <u>not closed</u>

<u>not convex</u>

<u>8.1</u> a) $\boxed{\max f = x - 2y + z \text{ when } x^2+y^2+z^2 \leq 3}$ <u>KT prob. Std. form</u>

$\mathcal{L} = x - 2y + z - \lambda(x^2+y^2+z^2-3)$

<u>FOC</u>: $\mathcal{L}'_x = 1 - \lambda\cdot 2x = 0$ <u>C</u>: $x^2+y^2+z^2 \leq 3$ <u>CSC</u>: $\lambda \geq 0$

$\mathcal{L}'_y = -2 - \lambda\cdot 2y = 0$ $\quad \lambda\cdot(x^2+y^2+z^2-3) = 0$

$\mathcal{L}'_z = 1 - \lambda\cdot 2z = 0$

$\Downarrow$

$x = \frac{1}{2\lambda}$ $\quad (\lambda \neq 0) \overset{CSC1}{\Rightarrow} \lambda > 0 \overset{CSC2}{\Rightarrow} x^2+y^2+z^2 = 3$

$y = \frac{-2}{2\lambda}$ $\quad \frac{1^2}{(2\lambda)^2} + \frac{(-2)^2}{(2\lambda)^2} + \frac{1^2}{(2\lambda)^2} = 3 \quad |\cdot(2\lambda)^2$

$z = \frac{1}{2\lambda}$ $\quad \frac{6 = 1+4+1 = 3\cdot(2\lambda)^2}{3}$

$(x,y,z;\lambda) = \underline{\left(\tfrac{1}{2}\sqrt{2}, -\sqrt{2}, \tfrac{1}{2}\sqrt{2}; \tfrac{\sqrt{2}}{2}\right)}$ $\quad (2\lambda)^2 = 2$

$2\lambda = \pm\sqrt{2} \quad \lambda = \pm\frac{\sqrt{2}}{2} \overset{CSC'}{=} \underline{\frac{\sqrt{2}}{2}}$

$\frac{1}{\lambda} = \frac{2}{\sqrt{2}} = \sqrt{2}$

<u>Conclusion</u>:

$x^2+y^2+z^2 \leq 3 \overset{EVT}{\Rightarrow}$ there is a max $\Rightarrow$ <u>List of candidate pts</u>:

<u>compact set</u>

$-\sqrt{3} \leq x, y, z \leq \sqrt{3}$

(1) $\left(\tfrac{1}{2}\sqrt{2}, -\sqrt{2}, \tfrac{1}{2}\sqrt{2}; \tfrac{\sqrt{2}}{2}\right)$ $f = 3\sqrt{2}$

(2) Adm. pts. where NDCQ fails: <u>none</u>

<u>Check</u>: <u>$x^2+y^2+z^2 = 3$</u>: $\operatorname{rk}(2x\ 2y\ 2z) = 1$ fails $\Rightarrow (0,0,0) \Rightarrow$ not possible.

$x^2+y^2+z^2 < 3$: no cond.

$\Downarrow$

$f_{max} = \underline{\underline{3\sqrt{2}}}$

b) max $f = xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$ KT-prob std. form

$\mathcal{L} = xz + yw - \lambda_1(x^2+y^2-1) - \lambda_2(4z^2+9w^2-36)$

FOC: $L'_x = z - \lambda_1 \cdot 2x = 0$
$L'_y = w - \lambda_1 \cdot 2y = 0$
$L'_z = x - \lambda_2 \cdot 8z = 0$
$L'_w = y - \lambda_2 \cdot 18w = 0$

C: $x^2+y^2 \le 1$
$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$
$\lambda_1(x^2+y^2-1) = 0$
$\lambda_2 \ge 0$
$\lambda_2(4z^2+9w^2-36) = 0$

FOC1: $z = 2\lambda_1 x$

FOC3: $x - 8\lambda_2(2\lambda_1 x) = 0$
$x(1 - 16\lambda_1\lambda_2) = 0$
$x = 0$ or $\lambda_1\lambda_2 = 1/16$

FOC2: $w = 2\lambda_1 y$

FOC4: $y - \lambda_2 \cdot 18(2\lambda_1 y) = 0$
$y(1 - 36\lambda_1\lambda_2) = 0$
$y = 0$ or $\lambda_1\lambda_2 = 1/36$

FOC: (a) $x=0,\ y=0$
(b) $x=0,\ \lambda_1\lambda_2 = 1/36$
(c) $\lambda_1\lambda_2 = 1/16,\ y=0$
~~(d) $\lambda_1\lambda_2 = 1/16,\ \lambda_1\lambda_2 = 1/36$~~

(a) $x=0, y=0$: $z=0,\ w=0 \Rightarrow \binom{NB}{NB} \overset{CSC}{\Rightarrow} \lambda_1 = \lambda_2 = 0$

$\Rightarrow (0,0,0,0;0,0)$ $f=0$

(b) $x=0,\ \lambda_1\lambda_2 = 1/36$: $z=0$
$w = 2\lambda_1 y$

$\lambda_1,\lambda_2 > 0 \Rightarrow \binom{B}{B}$

$x^2+y^2=1$ $y^2=1$
$4z^2+9w^2=36$ $9w^2=36$
$w^2=4$

$y=\pm 1$
$w = \pm 2$

$\Rightarrow (0,1,0,2;1,1/36)$ $f=2$ $\Rightarrow \lambda_1 = \dfrac{w}{2y}$

~~$(0,1,0,-2;-1,-1/36)$~~
~~$(0,-1,0,2;-1,-1/36)$~~
$(0,-1,0,-2;1,1/36)$ $f=2$

$\lambda_1 \ge 0 \quad \lambda_2 \ge 0$

c) $\lambda_1\lambda_2 = 1/16,\ y=0$: $\lambda_1, \lambda_2 > 0 \Rightarrow$ (B) $\lambda_1 > 0$, $\lambda_2 \geq$

$w = 0$, $z = 2\lambda_1 x$ $\quad x = \pm 1 \quad x^2 = 1 \quad x^2+y^2=1$

$\lambda_1 = \frac{z}{2x} \quad z = \pm 3 \quad 4z^2 = 36 \quad 4z^2+9w^2=36$

$\Rightarrow (1, 0, 3, 0; 3/2, 1/24) \quad f = 3$

~~$(1, 0, -3, 0; -3/2, -1/24)$~~

~~$(-1, 0, +3, 0; -3/2, -1/24)$~~

$(-1, 0, -3, 0; 3/2, 1/24) \quad f = 3$

$f_{max} = 3$?

$h(x,y,z,w) =$

SOC: $L(x,y,z,w; 3/2, 1/24) = xz + yw - \frac{3}{2}(x^2+y^2-1) - \frac{1}{24}(4z^2+9w^2-36)$

$= \underbrace{-\frac{3}{2}x^2 - \frac{3}{2}y^2 - \frac{1}{6}z^2 - \frac{9}{24}w^2 + xz + yw} + \frac{3}{2} + \frac{3}{2}$

$+3$

$$H(h) = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/4 \end{pmatrix} \quad \begin{matrix} D_1 = -3 \\ D_2 = 9 \\ D_3 = -3 \cdot 0 = 0 \\ \Delta_2 = 9/4 - 1 = 5/4 \end{matrix}$$

$\Delta_1 = -3, -3, -1/3, -3/4 < 0$

$\Delta_2 = 9, 1, 1/4, 0, 5/4, 9/4 \geq 0$

$\Delta_3 = 0, -5/12, *, * \leq 0$

$\Delta_4 = * = 0 \geq 0$

$$\begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3/4 \end{vmatrix} = -3(9/4 - 1) = -3 \cdot 5/4 = -15/4 < 0$$

$\Downarrow$

h concave

($H(h)$ neg. semidef.)

$\Downarrow$ SOC

$\underline{\underline{f_{max} = 3}}$ at $(1,0,3,0)$, $(-1,0,-3,0)$ $\quad \lambda_1 = 3/2$, $\lambda_2 = 1/24$

KT) std. form

9.1

a) max $f = 2x^2 - 4y^2 - 2z^2$ wh $x^4 + y^4 + z^4 \le 16$

$$L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - 16)$$

FOC:

$$L'_x = 4x - \lambda \cdot 4x^3 = 0 \qquad x^4 + y^4 + z^4 \le 16 \qquad \lambda \ge 0$$

$$L'_y = -8y - \lambda\, 4y^3 = 0 \qquad \lambda(x^4 + y^4 + z^4 - 16) = 0$$

$$L'_z = -4z - \lambda \cdot 4z^3 = 0$$

$$4x(1 - \lambda \cdot x^2) = 0 \qquad x = 0 \text{ or } x^2 = 1/\lambda$$

$$4y(-2 - \lambda \cdot y^2) = 0 \qquad y = 0 \text{ or } \cancel{y^2 = -2/\lambda}$$

$$4z(-1 - \lambda z^2) = 0 \qquad z = 0 \text{ or } \cancel{z^2 = -1/\lambda}$$

$\Downarrow$

$y = 0, z = 0 \to x = 0 \Rightarrow \lambda = 0 \Rightarrow (0, 0, 0; 0)$ $f = 0$

$\searrow$ $x \ne 0$, $x^2 = 1/\lambda$; $\lambda > 0$ B; $x^4 = 16$, $x^2 = 4$, $x = \pm 2$

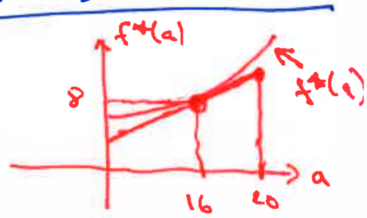
$\Downarrow$

$(\pm 2, 0, 0; 1/4)$ $f = 8$

$f_{max} = 8$ at $(\pm 2, 0, 0)$ with $\lambda = 1/4$

since $\{(x,y,z) : x^4 + y^4 + z^4 \le 16\}$ is compact

+ no adm. pts where NDCQ fails.



b) i) C $\to$ $x^4 + y^4 + z^4 \le 20$: max f wh $x^4 + y^4 + z^4 \le a$

$a = 16$: $f^*(16) = 8$ $x^*(16) = \pm 2$ $y^*(16) = 0$, $z^*(16) = 0$, $\lambda^*(16) = 1/4$

$a = 20$: $f^*(20) \cong \underbrace{f^*(16)}_{=8} + \underbrace{\Delta a}_{=(20-16)} \cdot \frac{df^*(a)}{da}$ Env. Thm.

$$= L'_a(x^*(a); \lambda^*(a))$$

$$= \lambda^*(a) = \lambda^*(16) = 1/4$$

$L = f(x,y,z) - \lambda \cdot (x^4 + y^4 + z^4 - a) = \ldots + \lambda a$

$L'_a = \lambda$

$$= 8 + 4 \cdot 1/4 = 9$$

ii) $f \to b x^2 - 4y^2 - 2z^2$: max $f = bx^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \le 16$

$\underline{b=2}$: $f^*(2) = \underline{8}$ $x^*(2) = \underline{\pm 2}$ $y^*(2) = z^*(2) = \underline{0}$ $\lambda^*(2) = \underline{1/4}$

$\underline{b=1}$: $f^*(1) \cong \underbrace{f^*(2)}_{8} + \underbrace{\Delta b}_{(1-2)} \cdot \dfrac{df^*(b)}{db}$

$\dfrac{df^*(b)}{db}$ = Env. Thm. $L'_b(\underline{x}^*(b); \underline{\lambda}^*(b)) = x^*(b)^2 = 4$

$= 8 - 1 \cdot 4 = \underline{\underline{4}}$

$L = \underline{bx^2} - 4y^2 - \dots$

$L'_b = x^2$

iii) $\underline{\text{Change } a \to 20 \text{ and } b = 1}$:

$f^*(20,1) \approx \underbrace{f^*(16,2)}_{8} + \underbrace{\Delta a}_{4} \cdot \underbrace{\dfrac{df^*(a)}{da}}_{1/4} + \underbrace{\Delta b}_{-1} \cdot \underbrace{\dfrac{df^*(b)}{db}}_{(\pm 2)^2}$

$= 8 + 4 \cdot 1/4 - 1 \cdot 4 = \underline{\underline{5}}$

9.3 $\max f(x,y,z,w)$ when $x^2+y^2+z^2+w^2=6$

$$f = \underline{x}^T A \underline{x} \qquad \underline{x}^T \cdot \underline{x} = 6, \quad \underline{x}^T\underline{x} = \underline{x}^T I \underline{x}$$

b) Find solutions of the Lagrange cond. with $\lambda = -12$

$$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T\underline{x} - 6)$$

FOC: $L'_{\underline{x}} = 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0} \;|:2 \Rightarrow A\underline{x} - \lambda I\underline{x} = \underline{0}$

C: $\underline{x}^T\underline{x} = 6$ $\qquad \boxed{(A-\lambda I)\underline{x} = \underline{0}}$

$\|x\|^2 = 6$ $\qquad |A-\lambda I| = 0$ or $\cancel{\underline{x} = \underline{0}}$

λ is eigenvalue and $\underline{x} \in E_\lambda$.

$\lambda = -12$:

$$A + 12I = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix} + \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 8 & 0 & 2 & 2 \\ 2 & 2 & 3 & 7 \end{pmatrix} \begin{matrix} -4 \\ \\ \\ -1 \end{matrix}$$

$$\to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 8 & -26 & -10 \\ 0 & 4 & -4 & 4 \end{pmatrix} \begin{matrix} \\ -4 \\ -2 \\ \end{matrix} \to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -18 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

w free, $z = -w$, $2y = 2z - 2w = -4w \Rightarrow y = -2w$

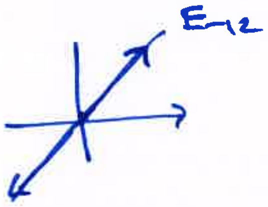
$2x = 2(-2w) - 7(-w) - 3w = 0 \Rightarrow x = 0$

$\Rightarrow (x,y,z,w) = (0,-2w,-w,w) = w \cdot (0,-2,-1,1)$

$\|(0,-2,-1,1)\|^2 = 0^2 + (-2)^2 + (-1)^2 + 1^2 = 6$

$\underline{w = \mathrm{I}\backslash}:$ $\quad (x, y, z, w; \lambda) = (0, -2, -1, 1; -12),$

$\qquad\qquad\qquad\qquad (0, 2, 1, -1; -12)$



c) $\underline{\text{SOC}}$: $h = \mathcal{L}(x, y, z, w; -12) = \underbrace{\underline{x}^T A \underline{x} + 12(\underline{x}^T \underline{x} - 6)}$

$= \underline{x}^T (A + 12I) \underline{x} - 72$

$H(h) = 2(\underbrace{A + 12I})$

matrix from the previous page

$$\begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix}$$

$D_1 = 8$

$D_2 = 16$

$D_3 = 8 \cdot 10 + 2 \cdot (-4) = 72$ ✓

$\underline{rk = 3}$ → RCC: pos. semidef.

$f_{min} = f(0, -2, -1, 1)$ $\quad \overset{\text{SOC}}{\Longleftarrow}$ h convex $\Longleftarrow$ pos. semidef.

$= \underline{x}^T A \underline{x}$

$= \underline{x}^T (-12 \underline{x})$

$= -12\, \underline{x}^T \underline{x} = -12 \cdot 6 = \underline{\underline{-72}}$

Exam Problems from Plenary Session 3

1) Final exam 11/2019 Q4
2) – " – 06/2018 Q4
3) Key Problem 9.3

① Final exam 11/2019, Q4

$\min f(x,y,z) = \underbrace{x^2+y^2+z^2-xy+xz-yz}_{\underline{x}^T A \underline{x}}$ when $x+y+z=11$ Lagrange

a)

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix} \qquad H(f) = 2A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

pos. def.

$D_1 = 2$
$D_2 = 3$
$D_3 = 1\cdot(-1) - (-1)\cdot(-1) + 2\cdot 3$
$= -1 - 1 + 6 = 4$

f convex

b) $\mathcal{L} = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 11)$ $\qquad B = (1\ 1\ 1)$

FOC: $\mathcal{L}'_{\underline{x}} = \boxed{\begin{matrix} 2A\cdot\underline{x} - \lambda\cdot B^T = \underline{0} \\ B\underline{x} = 11 \end{matrix}}$

C:

$A\underline{x} = \frac{1}{2}\lambda B^T \Rightarrow \underline{x} = A^{-1}(\frac{1}{2}\lambda B^T)$

$\underline{x} = \frac{\lambda}{2}\cdot A^{-1}B^T$

$B\underline{x} = B\cdot \lambda/2 \cdot A^{-1}B^T = 11$

$\lambda/2 \cdot \underline{BA^{-1}B^T} = 11$

$\Uparrow$

FOC:
$$\begin{aligned} 2x - y + z - \lambda\cdot 1 &= 0 \\ -x + 2y - z - \lambda\cdot 1 &= 0 \\ x - y + 2z - \lambda\cdot 1 &= 0 \end{aligned}$$
C:
$$x + y + z = 11$$

4x4 linear system
$\Downarrow$
use Gauss

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 2 & -1 & 1 & -1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \end{array}\right) \begin{matrix} -2 \\ 1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -3 & -1 & -1 & -22 \\ 0 & 3 & 0 & -1 & 11 \\ 0 & -2 & 1 & -1 & -11 \end{array}\right) -1$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 3 & 0 & -1 & 11 \\ 0 & -2 & 1 & -1 & -11 \end{array}\right) \begin{matrix} 3 \\ -2 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -6 & -1 & -22 \\ 0 & 0 & 5 & -1 & 11 \end{array}\right) 9,$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -1 & -2 & -11 \\ 0 & 0 & 5 & -1 & 11 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -1 & -2 & -11 \\ 0 & 0 & 0 & -11 & -44 \end{array}\right)$$

$-11\lambda = -44 \qquad \underline{\lambda = 4}$

$-z - 8 = -11 \qquad -z = -3 \qquad \underline{z = 3}$

$-y - 2\cdot 3 = -11 \qquad -y = -5 \qquad \underline{y = 5}$

$x + 5 + 3 = 11 \qquad \underline{x = 3}$

Cand. pt:

$(x, y, z; \lambda) = \underline{(3, 5, 3; 4)}$

$f = 9 + 25 + 9 - 15 + 9 - 15 = \underline{22}$

SOC: $h = f(x,y,z) - \underline{4(x+y+z-11)}$

$H(h) = H(f)$ pos. defn. from a) $\Rightarrow h$ convex $\overset{\text{SOC}}{\Rightarrow}$ $f_{\min} = \underline{\underline{22}}$ at $(3,5,3)$ with $\lambda = 4$

c) Change C: $\underline{\underline{x+y+z=10}}$:

min $f(x,y,z)$ when $x+y+z=a$

$L = f(x,y,z) - \lambda\,(x+y+z-a)$

$\Rightarrow L'_a = \underline{\lambda}$

$\xrightarrow{\text{Env. Thm}} \quad \dfrac{df^*(a)}{da} = L'_a\left(\underline{x}^*(a); \lambda^*(a)\right) = \lambda^*(a) = 4$ when $a = 11$

$f^*(10) \simeq f^*(11) + \Delta a \cdot \dfrac{df^*(a)}{da} = 22 + (-1)\cdot 4 = \underline{\underline{18}}$

2 Final exam 06/2018, Q4

KT-prob. std. form

max $f(x,y) = \underline{xy(x-y)}$ when $x^2 + y^2 + (x-y)^2 \le 96$

a) $L = xy(x-y) - \lambda\left(x^2 + y^2 + (x-y)^2 - 96\right)$
$\quad = x^2y - xy^2 - \lambda\,(x^2 + y^2 + x^2 - 2xy + y^2 - 96)$

$$\left.\boxed{\begin{array}{ll} \textbf{FOC:}\ L'_x = & 2xy - y^2 - \lambda\cdot(2x + 2x - 2y) = 0 \\ \phantom{\textbf{FOC:}}\ L'_y = & x^2 - 2xy - \lambda\,(2y - 2x + 2y) = 0 \\ \textbf{C:} & x^2 + y^2 + (x-y)^2 \le 96 \\ \textbf{CSC:} & \lambda \ge 0 \text{ and } \lambda\cdot\left(x^2 + y^2 + (x-y)^2 - 96\right) = 0 \end{array}}\right\} \text{KT-conditions}$$

b) Find all $(x;y;\lambda)$ satisfying FOC + C + CSC with $\underline{\underline{x,y \neq 0}}$

$\underline{\text{FOC:}}$ $y(2x-y) - \lambda(4x-2y) = 0$

$x(x-2y) - \lambda(-2x+4y) = 0$

↙ NB ↘ B

$\underline{x^2+y^2+(x-y)^2 < 96}$:

$\underline{\text{CSC:}}$ $\lambda = 0$

$y(2x-y) = 0$

$x(x-2y) = 0$

$\Downarrow$

$2x-y=0$

$x-2y=0$

$\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$

just trivial solution $x=y=0$

$\underline{\text{concl:}}$ $\boxed{\text{no cand. pts with } x,y \neq 0}$

$\underline{x^2+y^2+(x-y)^2 = 96}$:

$\underline{\text{CSC:}}$ $\lambda \geq 0$

$\underline{\text{FOC:}}$ (1) $\lambda = \dfrac{y(2x-y)}{4x-2y}$ or $\underline{2x-y=0}$

$= \dfrac{y\cancel{(2x-y)}}{2\cancel{(2x-y)}}$

$\underline{\lambda = y/2}$

(2) $\lambda = \dfrac{x(x-2y)}{-2x+4y}$ or $\underline{x-2y=0}$

$= \dfrac{x\cancel{(x-2y)}}{-2\cancel{(x-2y)}}$

$\underline{\lambda = -\frac{x}{2}}$

$\underline{\text{Cases:}}$

(A) $\underline{\lambda = y/2 = -x/2}$: $y=-x$

$x^2+(-x)^2+(2x)^2 = 96$

$6x^2 = 96$

$x^2 = 96/6 = 16$

$x = \pm 4$ $y = \mp 4$ $\lambda = \mp 2$

$\Rightarrow$ $\underline{\text{Cand:}}$ $(4,-4;-2)$ $f=-128$

$(-4,4;2)$ $f=128$

(B) $\underline{\lambda = y/2,\ x=2y}$:

$(2y)^2+y^2+y^2 = 96$

$6y^2 = 96$

$y^2 = 16$

$y=\pm 4$ $x = \pm 8$; $\lambda = \pm 2$

$\Rightarrow$ $\underline{\text{Cand pts:}}$ $(8,4;2)$ $f=128$

$(-8,-4;-2)$ $f=-128$

B (cont'd)

(C) $\underline{2x-y=0,\ \lambda = -x/2}$:

$y=2x \rightarrow x^2+(2x)^2+(-x)^2 = 96$

$6x^2 = 96$

$x^2 = 16$

$\lambda = \mp 2$ $y = \pm 8$ $x = \pm 4$

$\underline{\text{Cand. pts:}}$ $(4,8;-2)$ $f=-128$

$(-4,-8;2)$ $f=128$

(D) $\underline{2x-y=0,\ x-2y=0}$:

$x=y=0 \Rightarrow 0+0+0 \neq 96$

$\Rightarrow$ no points

$\underline{\text{Concl:}}$ six points from

$\underline{\text{B}}$, case (A), (B), (C)

(c) $C: x^2+y^2+(x-y)^2 \le 96$ compact set since $x^2 \le 96$, $y^2 \le 96$

$\Leftarrow$ EVT $\quad\Leftarrow$

there is a maximum

$-\sqrt{96} \le x \le \sqrt{96}$
$-\sqrt{96} \le y \le \sqrt{96}$

$f_{max} = 128$ since the possible candidate pts are

ordinary cand. pts

(i) pts $(x;y;\lambda)$ with $x \cdot y \ne 0$ that satisfies FOC+C+CSC
$\Rightarrow$ six points, max f-value $= 128$
(b)

(ii) pts $(x;y;\lambda)$ with $x=0$ or $y=0$ — " —
$\Rightarrow$ any such point will have $f=0$, **not** max points

iii) Adm points where NDCQ fails: no points

$C: x^2+y^2+(x-y)^2 \le 96$

$\to$ B, NDCQ: $\mathrm{rk}\left(\underbrace{2x+2(x-y)}_{g'_x} \quad \underbrace{2y+2(x-y)(-1)}_{g'_y}\right)=1$

$\to$ NB no NDCQ ok

NDCQ fails:

$g'_x = g'_y = 0$

$4x-2y=0$, $-2x+4y=0$

$y=2x$ $\quad -2x+8x=0$

$6x=0$

$y=0$ $\quad x=0$

$\Downarrow$

$(0,0)$ $\quad 0+0+0<96$

not B

③ Key Problem 9.3

max $f(x,y,z,w) = \underline{x}^T A \underline{x}$ when $\underline{x}^T \underline{x} = 6$

d) $L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T \underline{x} - 6)$ ← $\underline{x}^T \cdot I \underline{x}$

FOC
$L'(\underline{x}) = 2A\underline{x} - \lambda \cdot (2\underline{x}) = 0$ ← $2I\underline{x}$

$2 \cdot [(A - \lambda I)\underline{x}] = 0$

$(A - \lambda I)\underline{x} = 0$

$\cancel{\underline{x} = 0}$ or $|A - \lambda I| = 0$
λ eigenvalue of A
$\underline{x} \in E_\lambda$ eigenvector

C $\underline{x}^T \underline{x} = 6$

$\Downarrow$

Cond pts: $(\underline{x}; \lambda)$ s.t. λ is eigenvalue of A
$\underline{x}$ " eigenvector of A
with $\underline{x}^T \underline{x} = 6$

$f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \cdot (\lambda \underline{x})$

$= \lambda \cdot \underline{x}^T \underline{x} = \lambda \cdot 6 = \underline{6\lambda}$ ← Notice: the highest eigenvalue gives the highest f-val.

From (a): f concave and
H(f) negative semidefin.

$\Updownarrow$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 0$ and at least one $\lambda_i = 0$ $\Big\}$ $\Rightarrow$ the highest eigenvalue $= 0$

$f_{max} = \underline{\underline{0}}$

a) $f(x,y,z,w) = -4x^2 - 10y^2 - 5z^2 - 5w^2$
$\quad + 4xz + 4xw - 4yz + 4yw + 6zw$

$$A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix}$$

$D_1 = -4 > 0$

$D_2 = 40 < 0$

$D_3 = -4 \cdot (50-4) + 2(0-(-20))$
$\quad = -4 \cdot 46 + 2 \cdot 20 = -184 + 40 = -144 < 0$

$D_4 = 0$ since $R(4) = -R(3)$

$$\begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix} \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \rightarrow \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 0 & -2 & -4 & 4 \\ 0 & 2 & 4 & -4 \end{pmatrix}$$

$\Rightarrow$ rk $A = 3$, RRC: A <u>neg. semidefinite</u>