	Plan	
1	Key Problems:	5.1b, 5.5bd, 6.1, 6.3d, 6.4be, 6.5
2	Midterm Exams:	01/2020 Q4,5,7 ; 10/2019 Q7,8 ; 01/2019 Q5,7 ; 10/2018 Q2,3,8 ; 05/2018 Q2,5 ; 01/2018 Q5,7,8 ; 10/2017 Q3,5 ; 05/2017 Q5 ; 01/2017 Q8 ; 01/2010 Q5

2 Midterm exams

01/2020

4

$$A = \left(\boxed{\begin{matrix} 3 & 6 \\ 4 & t \end{matrix}} \; \boxed{\begin{matrix} -1 & -t \\ -t & 1 \end{matrix}} \right) \qquad rk(A) = \begin{cases} 2, & t \neq 8 \\ 2, & t = 8 \end{cases} = \underline{2}$$

$$M_{12,12} = 3t - 24 = 0$$
$$\underline{t = 8}$$

$$M_{12,34} = -1 - t^2 \neq 0 \quad \text{when } t = 8 \qquad \text{(D)}$$

5.

$$A = \begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.7 \end{pmatrix} \qquad \underline{\text{regular}}$$

$\underline{\lambda = 1:}$

$$\begin{pmatrix} -0.6 & 0.2 & 0.2 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.3 \end{pmatrix} \begin{matrix} \cdot 10 \\ \cdot 10 \\ \cdot 10 \end{matrix} \to \begin{pmatrix} 2 & 2 & -3 \\ 4 & -4 & 1 \\ -6 & 2 & 2 \end{pmatrix} \begin{matrix} \\ -2 \\ 3 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 2 & -3 \\ 0 & -8 & 7 \\ \cancel{0} & \cancel{8} & \cancel{-7} \end{pmatrix} \qquad \begin{aligned} 2x + 2y - 3z &= 0 \\ -8y + 7z &= 0 \end{aligned} \qquad y = \frac{7z}{8}$$

$$\frac{2x}{2} = 3z - 2\left(\frac{7z}{8}\right) = z \cdot \left(\frac{24}{8} - \frac{14}{8}\right) = \frac{\frac{10}{8}z}{2} \qquad x = \frac{10}{16} z = \frac{5}{8} z$$

$x = \frac{5}{8}z \quad y = \frac{7}{8}z \quad z = z$

$x+y+z=1$

$\frac{5}{8}z+\frac{7}{8}z+z=1$

$z \cdot \left(\frac{20}{8}\right)=1$

$\underline{z = \frac{8}{20}}$

$V_1 = x = \frac{5}{8} \cdot \frac{8}{20} = \frac{5}{20} = \underline{25\%}$

D

$\underline{7}.\quad f = x^4+y^4+z^4-4xy$

$f'_x = 4x^3-4y = 0$

$f'_y = 4y^3-4x=0$

$f'_z = 4z^3=0$

$\underline{y=x^3}$

$\underline{x=y^3}$

$\underline{z=0}$

$x=(x^3)^3=x^9$

$x-x^9=0$

$x(1-x^8)=0$

$x=0$ or $x^8=1$

$y=0 \quad x=\pm 1$

$z=0$

$\downarrow$

$\underline{(0,0,0)}$

$x=1 \quad x=-1$

$y=1 \quad y=-1$

$z=0 \quad z=0$

$\downarrow$

$\underline{(1,1,0)} \quad \underline{(-1,-1,0)}$

$$H(f) = \begin{pmatrix} 12x^2 & -4 & 0 \\ -4 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

(a) $H(f)(0,0,0) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$D_2 = -16$ indef. $\underline{\text{saddle pt}}$

(b) $H(f)(1,1,0) = H(f)(-1,-1,0) = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$D_1 = 12$

$D_2 = 128$

$D_3 = 0$

pos. semidefn. (KKC)

SOT $\underline{\text{inconclusive}}$

$\underline{\text{Alt methods}}$: 1) f convex/concave? $\underline{\text{no}}$.

2) $\underline{\text{Defn}}$: $f(1,1,0) = -2$ $\underline{\text{local min}}$

$f(x,y,z) = \underbrace{x^4+y^4-4xy}_{(1,1)\text{ loc. min}} + \underbrace{z^4}_{z=0\text{ is min}}$

B

10/2019

7) $f = x^3 + y^3 + z^3 - 3xz$

$$f'_x = 3x^2 - 3z = 0 \qquad x^2 = z$$
$$f'_y = 3y^2 = 0 \qquad y = 0$$
$$f'_z = 3z^2 - 3x = 0 \qquad z^2 = x$$

$\Rightarrow \; x = (x^2)^2 = x^4$

$x - x^4 = 0$

$x(1 - x^3) = 0$

$x = 0$ or $x^3 = 1$

$z = 0$, $y = 0$ $\Rightarrow$ $\underline{(0,0,0)}$ | $x = 1$, $z = 1$, $y = 0$ $\Rightarrow$ $\underline{(1,0,1)}$

$$H(f) = \begin{pmatrix} 6x & 0 & -3 \\ 0 & 6y & 0 \\ -3 & 0 & 6z \end{pmatrix}$$

i) $H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}$ $\Delta_2 = -9$ Indef. saddle pt.

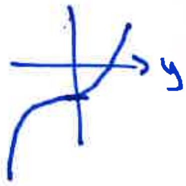
ii) $H(f)(1,0,1) = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{pmatrix}$ $D_1 = 6$, $D_2 = 0$, $D_3 = 0$; $\Delta_1 = 6, 0, 6$, $\Delta_2 = 0, 0, 27$, $\Delta_3 = 0$ pos. semid.

Alt. methods: i) convex/concave: no

ii) Defn: $f(1,0,1) = -1$

$f(1,y,1) = \underline{-1 + y^3}$

saddle pt.



$f = \underbrace{x^3 + z^3 - 3xz} + \underbrace{y^3}$

8). dim Null(A) = # vectors in a base = 2
"
free variables

(D)

01/2019.

5.) $A = \begin{pmatrix} 1 & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & 1 \end{pmatrix}$

i) # eigenvalues = 3

ii) # eigenvectors = 3 (lin. indep.)

$$\begin{vmatrix} 1-\lambda & 0 & -s \\ 0 & 1-\lambda & 0 \\ s & 0 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)\cdot\left[(1-\lambda)^2 + s^2\right] = 0$$

$\lambda = 1$ or $(1-\lambda)^2 + s^2 = 0$

$s = 0$: $\lambda_2 = \lambda_3 = 1$ $\quad$ $s \neq 0$: no soln.

$s = 0$: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (D)

7) $f = 5x^2 + 4xy + y^2 + 3z^2 + 2zw + w^2$

$$A = \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} x \\ y \\ z \\ w \end{array}$$

$D_1 = 5$

$D_2 = 1$

$D_3 = 3 \cdot 1 = 3$

$D_4 = 1 \cdot D_3 - (1) \cdot \begin{vmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 3 - (1 \cdot D_2) = 2$

pos. defn.

(B)

10/2018.

2).
$$\begin{pmatrix} \boxed{t} & 3 \\ \boxed{2} & 6 \\ 3 & t \\ 5 & 9+t \end{pmatrix}$$

$M_{12,12} = 6t - 6 = 0 \qquad t = 1$

$M_{23,12} = 2t - 18 = -16 \neq 0 \quad (t=1)$

$t \neq 1$: lin. independent

$t = 1$: — " —

(A)

3.)
$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 4 & 0 & 6 \\ t & -1 & 5 & 3 \end{pmatrix} \begin{matrix} -2 \\ -t \end{matrix}$$

$$\to \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & -1-3t & 5+t & 3-4t \end{pmatrix} : 2$$

$$\to \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -1 & 1 & -1 \\ 0 & -1-3t & 5+t & 3-4t \end{pmatrix} -1-3t$$

$$\to \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & (4-2t) & (4-t) \end{pmatrix}$$

$t \neq 2 \qquad t = 2$

(A)

8) $f(x,y,z) = 1-(x-y+z)^4 = \underline{1-u^4}$, $u = x-y+z$

$\underline{f_{max} = 1}$ at $\underline{u=0}$ $(x,y,z)=(1,1,0) \to u=0$

$f'_x = -4u^3 \cdot 1 = \underline{-4u^3}$

$f'_y = -4u^3(-1) = \underline{4u^3}$

$f'_z = -4u^3 \cdot 1 = \underline{-4u^3}$

$$H(f) = \begin{pmatrix} -12u^2 \cdot 1 & 12u^2 & -12u^2 \\ 12u^2 & -12u^2 & 12u^2 \\ -12u^2 & 12u^2 & -12u^2 \end{pmatrix}$$

$D_1 = -12u^2 \leq 0$

$\left.\begin{matrix} D_2 = 0 \\ D_3 = 0 \end{matrix}\right\}$ $rk = 1$ $R(2) = -R(1)$, $R(3) = R(1)$

RRC: neg. semidefin.

$\Rightarrow$ $\underline{\text{concave}}$ (D)

05/2018:

2) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & t \end{pmatrix} \begin{matrix} \\ ]-1 \\ ]-1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & t-1 \end{pmatrix}]-3 \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \boxed{t-7} \end{pmatrix}$

$t \neq 7$: lin. independent. (D)

$t = 7$: lin. dependent

5) $A = \begin{pmatrix} 1 & s & -1 \\ 0 & 0 & s \\ 0 & 0 & 1 \end{pmatrix}$ $\lambda = 1, 0, 1$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\underline{\lambda = 1}$ $(m=2)$ need two free variables

$P = \Big(\underbrace{|\ \ |}_{\lambda=1} \ \underbrace{|}_{\lambda=0} \Big)$

$\begin{pmatrix} 0 & s & -1 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & s \\ 0 & s & -1 \\ 0 & 0 & 0 \end{pmatrix}]s \to \begin{pmatrix} 0 & -1 & s \\ 0 & 0 & s^2-1 \\ 0 & 0 & 0 \end{pmatrix}$ (A)

$s = \pm 1$; diag.

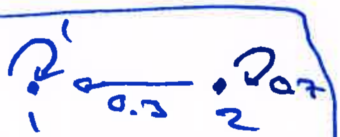
1 Key Problems

5.1 b) $A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$ $D_1 = -1$, $D_2 = 2$ neg. defn.

5.5 b) $A = \begin{pmatrix} 0.86 & 0.42 \\ 0.14 & 0.58 \end{pmatrix} > 0$ regular

$\lambda = 1$: $\begin{pmatrix} -0.14 & 0.42 \\ \cancel{0.14} & \cancel{-0.42} \end{pmatrix} \to t\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 0.3 \\ 0 & 0.7 \end{pmatrix}$



not regular

6.1 $f = x^2 + ay^2 - 2ayz + az^2 + w^2 - 2xw$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & a & -a & 0 \\ 0 & -a & a & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} x \\ y \\ z \\ w \end{array} \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array}$$

$D_1 = 1$
$D_2 = a$
$D_3 = 0$
$D_4 = 0$

$\operatorname{rk} A = \begin{cases} 2, & a \neq 0 \\ 1, & a = 0 \end{cases}$

$R(3) = -R(2)$ $R(4) = -R(1)$

a) $a > 0$:
$\operatorname{rk} A = 2$
$D_1, D_2 > 0$
RRC:
pos semid.

b) $a = 0$:
$\operatorname{rk} A = 1$
$D_1 > 0$
RRC:
pos. semid.

c) $a < 0$:
$D_2 = a < 0$
indefn.

63 d) $f = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

$$f'_x = -4x^3 - 4x + 6z = 0$$
$$f'_y = -6y = 0 \Rightarrow \underline{y = 0}$$
$$f'_z = 6x - 12z = 0 \Rightarrow \underline{x = 2z}$$

$$-4(2z)^3 - 4\cdot(2z) + 6z = 0$$
$$-32z^3 - 8z + 6z = 0$$
$$-32z^3 - 2z = 0$$
$$-2z(16z^2 + 1) = 0$$
$$\underline{z = 0} \Rightarrow \underline{x = 0}$$

$(x,y,z) = \underline{\underline{(0,0,0)}}$

$$H(f) = \begin{pmatrix} -12x^2 - 4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$\underline{(0,0,0)}$:

$$\begin{pmatrix} -4 & 0 & +6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix} \quad \begin{matrix} D_1 = \underline{-4} \\ D_2 = \underline{24} \end{matrix}$$

$$D_3 = -6\cdot(48 - 36) = \underline{-72}$$

neg. defn $\Rightarrow$ $\underline{\underline{\text{loc. max}}}$

$\underline{f \text{ concave?}}$ $\Leftrightarrow$ $H(f)(x,y,z)$ is neg. semidef. for all pts (x,y,z)

Yes

$$D_1 = -12x^2 - 4 \le -4 \quad \text{neg.}$$
$$D_2 = -6\cdot D_1 > 0 \quad \text{pos.}$$
$$D_3 = -6\cdot(144x^2 + 48 - 36)$$
$$= -6\,(\underbrace{144x^2 + 12}_{>0}) < 0 \quad \underline{\text{neg.}}$$

$H(f)$ is neg. semidefn.

f concave

$\Downarrow$

$(0,0,0)$ $\underline{\underline{\text{global max}}}$

6.5. Find $V(f)$:

a) $f(x,y,z) = \ln(1+2x^2+2xy+3y^2-2xz+z^2)$

$= \ln(1+u)$, $u = 2x^2+2xy+3y^2-2xz+z^2$

Inner: u

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$D_1 = 2$

$D_2 = 5$

$D_3 = 1\cdot 5 - 1\cdot 3 = 2$

pos. def.

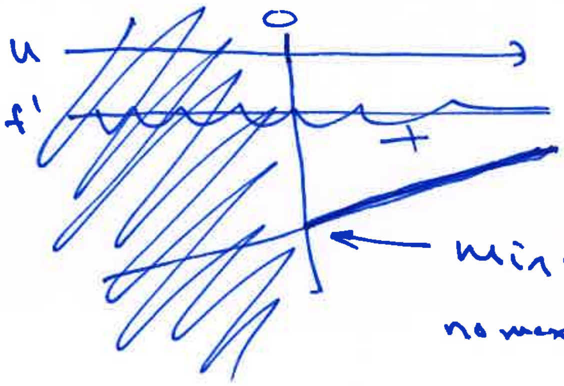
min $= 0$

$\boxed{u \geq 0}$

$V_u = [0, \to)$

Outer: $f = \ln(1+u)$, $u \geq 0$

$f' = \frac{1}{1+u}\cdot 1 = \frac{1}{1+u} > 0$



min: $f(0) = \ln(1) = 0$

no max: $\lim_{u\to\infty} \ln(1+u) = \infty$

$V_f = [0, \to)$

b) $f = (x^2+y^2+z^2)\cdot e^{-x^2-y^2-z^2}$

$= u\cdot e^{-u}$, $u = x^2+y^2+z^2$

Inner: u pos. def.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{u \geq 0}$

Outer fn.: $f(u) = ue^{-u}$

$f' = 1\cdot e^{-u} + u\cdot e^{-u}\cdot(-1)$

$= (1-u)\,e^{-u}$

$u=0$: $f = 0$

$u=1$: $f = 1\cdot e^{-1} = 1/e$

$u\to\infty$: $\lim_{u\to\infty} ue^{-u} = \lim_{u\to\infty} \frac{u}{e^u} = 0$

$V_f = [0, 1/e]$

2) 05/2017:

5) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & 1 \\ 0 & 0 & s-1 \end{pmatrix}$ $\quad \lambda = 1, s, s-1$

$s=1$: $\underline{1, 1, 0}$ $\quad$ $\underline{\lambda = 1 \ (m=2)}$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ ok.

$s=2$: $\underline{1, 2, 1}$: $\quad$ $\underline{\lambda = 1 \ (m=2)}$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ok.

$s \neq 1, 2$: $\boxed{\begin{matrix} 1, s, s-1 \\ \text{all mult. } 1 \end{matrix}}$: ok

A

8) $f = \sqrt{x^2+y^2+3} = \sqrt{u} = \underline{u^{1/2}}, \quad u = x^2+y^2+3$

$f'_x = \cancel{\tfrac{1}{2}} u^{-1/2} \cdot \cancel{2}x = \dfrac{x}{\sqrt{u}}$

⋮

$$f''_{xx} = \frac{1 \cdot \sqrt{u} - x \cdot \cancel{\tfrac{1}{2}} u^{-1/2} \cdot \cancel{2}x}{u} = \frac{\left(\sqrt{u} - x^2/\sqrt{u}\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \frac{u - x^2}{u\sqrt{u}} = \underline{\underline{\frac{y^2+3}{u\sqrt{u}}}}$$

⋮