

Plan

- 1 Introduction
- 2 Python: NumPy and ndarrays
- 3 Project: Coding the Gaussian process

① Introduction

Lecture 1-13
Fri 08-10
2+1 maths

(with GRAGORIS)

+

Lecture A-C
Wed 15-17
2+(maths + python)

(small class)

Exams: Final exam
(around Dec 02)
80%

Home exam
(midterm)
20%

② Python: NumPy and ndarrays

Book: Python for data analysis (2e)
For this lecture: Chapter 4

* NumPy: numerical Python package

```
import numpy as np
```

* ndarrays: data structure for n-dim. arrays

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{pmatrix}$$

```
A = np.array([[1,1,1,3],[1,2,4,7],[1,3,9,13]])
B = np.random.randn(3,5)
```

data types: A.dtype → np.int32 (integers)
B.dtype → np.float64 (floats = decimal no.)

rows, cols: A[1] → [1,2,4,7]
A[1,1] → 2
A[:,1] → $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

python: row/col no's start at 0

points,
not copies:

$$C = A$$

$$C = A.copy()$$

$$C[1,1] = 5$$

$$A \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 7 \\ 1 & 3 & 1 \end{pmatrix}$$

③ Project: Coding Gaussian process

Input: matrix (as an array)

Output: an echelon form

Functions: def f(x):
return (x+1)

$$f(3) \rightarrow 4$$

def fact(n):
if n==1:
return 1
return (n*fact(n-1))

$$\text{fact}(3) \rightarrow 3 \cdot \text{fact}(2)$$

$$\rightarrow 3 \cdot \underline{2 \cdot \text{fact}(1)}$$

$$\rightarrow 3 \cdot 2 \cdot 1 = 6$$

recursive fn.

Row operations:

def Rmult(matrix, i, c)

def Radd(matrix, i, j, c)

def Rswitch(matrix, i, j)

Note: i, j are
actual row numbers /
col. numbers

↓
i-1, j-1 in Python

```
In [12]: import numpy as np
```

```
In [13]: A=np.array([[1,1,1,3],[1,2,4,7],[1,3,9,13]])
```

```
In [14]: def Rmult(matrix,i,c):  
    matrix[i-1]=matrix[i-1]*c  
    return(matrix)
```

```
In [15]: def Radd(matrix,i,j,c):  
    matrix[j-1]=matrix[j-1]+matrix[i-1]*c  
    return(matrix)
```

```
In [16]: def Rswitch(matrix,i,j):  
    r = matrix[i-1].copy()  
    matrix[i-1]=matrix[j-1]  
    matrix[j-1]=r  
    return(matrix)
```

```
In [17]: A
```

```
Out[17]: array([[ 1,  1,  1,  3],  
                [ 1,  2,  4,  7],  
                [ 1,  3,  9, 13]])
```

```
In [18]: Radd(A,1,2,-1)
```

```
Out[18]: array([[ 1,  1,  1,  3],  
                [ 0,  1,  3,  4],  
                [ 1,  3,  9, 13]])
```

```
In [19]: Radd(A,1,3,-1)
```

```
Out[19]: array([[ 1,  1,  1,  3],  
                [ 0,  1,  3,  4],  
                [ 0,  2,  8, 10]])
```

```
In [20]: Radd(A,2,3,-2)
```

```
Out[20]: array([[1, 1, 1, 3],  
                [0, 1, 3, 4],  
                [0, 0, 2, 2]])
```

```
In [ ]: def Gauss(A):  
    ...
```

Plan

- 1 Complex numbers and roots of equations
- 2 Polar coordinates and de Moivre's formula

① Complex numbers and roots of equation

Ex: $x^2 - 4x + 5 = 0$

$$x^2 - 4x = -5$$

$$x^2 - 4x + 4 = -5 + 4$$

$$(x-2)^2 = -1$$

$$x = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2}$$

$$= \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm \sqrt{4} \cdot \sqrt{-1}}{2}$$

$$= \underline{2 \pm i}$$

$i = \sqrt{-1}$:

$$x-2 = \pm \sqrt{-1}$$

$$x-2 = \pm i$$

$$x = 2 \pm i$$

$$x_1 = \underline{2+i}, \quad x_2 = \underline{2-i}$$

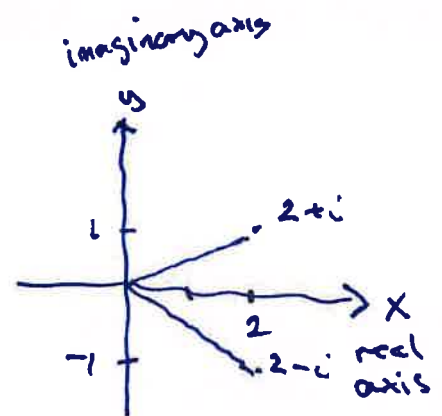
Complex number:

$$z = a + bi$$

where a, b are real numbers

↑

(a, b) in \mathbb{R}^2



Complex plane

Ex: Computations:

$$z_1 = 2 + 3i, \quad z_2 = 1 - i$$

$$z_1 + z_2 = 3 + 2i$$

$$z_1 - z_2 = 1 + 4i$$

$$z_1 \cdot z_2 = (2 + 3i) \cdot (1 - i) = 2 + 3i - 2i - 3i^2$$

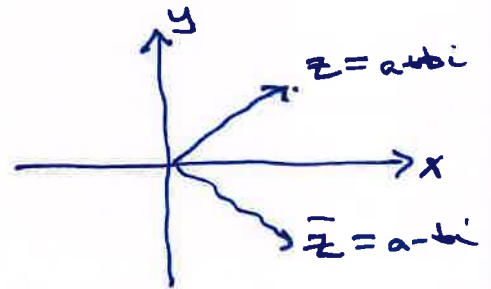
$$= \underline{5 + i}$$

$$i^2 = -1$$

Ex: $(1+i) \cdot (1-i) = 1 + \cancel{i} - \cancel{i} - i^2 = \underline{2}$

Defn: $z = a + bi$

$\bar{z} = a - bi$
 z conjugate



Formula:

$z \cdot \bar{z} = |z|^2$

$|z| = \sqrt{a^2 + b^2}$
 modulus of z

$(a+bi)(a-bi)$
 $= a^2 + \cancel{abi} - \cancel{abi} - b^2 \cdot i^2$
 $= a^2 + b^2$

Python:

$z = 1 + j$

$z = \text{complex}(1, 1)$

module: cmath

A: random matrix
 of real numbers

np.linalg.eig(A)

Division:

$\frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i+i+i^2}{2} = \frac{\cancel{1} + 2i + \cancel{1}}{2} = \underline{i}$

$\frac{1+i}{2+3i} = \frac{(1+i)(2-3i)}{(2+3i)(2-3i)} = \frac{2+2i-3i+3}{4+9} = \frac{5-i}{13}$

$= \frac{5}{13} + i \cdot \left(\frac{-1}{13}\right)$

Roots of equations:

Ex: $x^3 - 5x^2 + 17x - 13 = 0$ $x=1$ is a solution

$(x-1)(x^2 - 4x + 13) = 0$

$x=1$ or $x^2 - 4x + 13 = 0$

$x^2 - 4x + 4 = -9$

$(x-2)^2 = -9$

$x-2 = \pm\sqrt{-9} = \pm 3i$

$x = 2 \pm 3i$

Roots: $x_1 = 1$ $x_2 = 2 + 3i$, $x_3 = 2 - 3i$

Result: Any polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ with real coeff $a_n, a_{n-1}, \dots, a_0, a_1, a_0$ in \mathbb{R} and $a_n \neq 0$ has:

- i) n complex solutions
- ii) if z is a solution, then \bar{z} is a solution

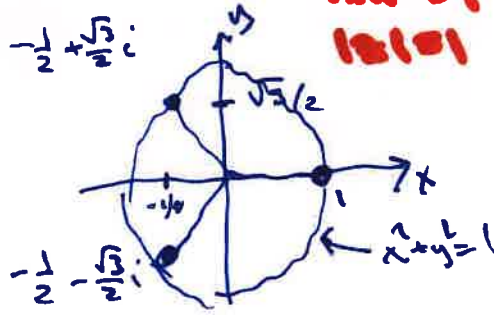
Ex: $x^3 = 1$ $x^3 - 1 = 0$ $x=1$ is a solution

$|z|^3 = 1$
 $|z| = 1$

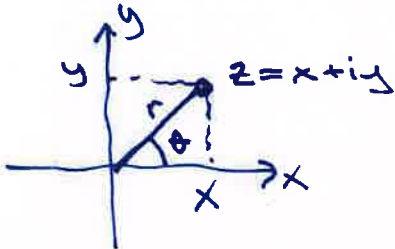
$(x-1) \cdot (x^2 + x + 1) = 0$
 $x=1$, $x = \frac{-1 \pm \sqrt{1-4 \cdot 1}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} \cdot i}{2}$

$x_1 = 1$, $x_2 = \frac{-1 + \sqrt{3}i}{2}$, $x_3 = \frac{-1 - \sqrt{3}i}{2}$



② Polar coordinates and de Moivre's formula



(x, y) : cartesian coordinates

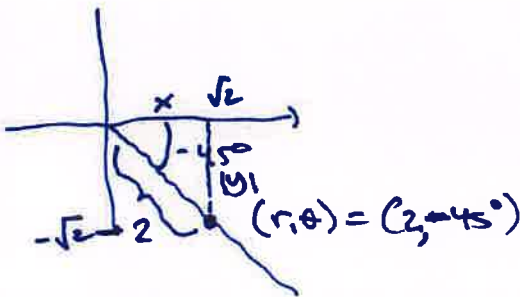
(r, θ) : polar coordinates

$$r = |z| = \sqrt{x^2 + y^2}$$

distance to $(0,0)$

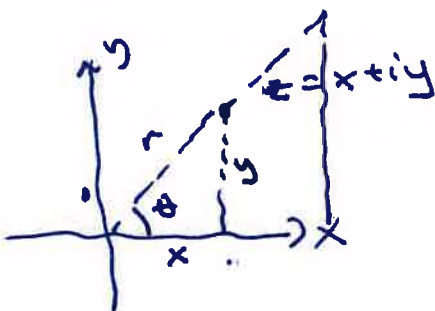
θ : angle with the positive x-axis (counterclockwise)

Ex:
 $(r, \theta) = (2, -45^\circ)$



$$\begin{aligned} x = |y| &\Rightarrow x^2 + y^2 = 2^2 \\ 2x^2 &= 4 \\ x^2 &= 2 \\ x = \sqrt{2}, y &= -\sqrt{2} \end{aligned}$$

$$\left. \begin{aligned} (r, \theta) &= (2, -45^\circ) = (2, 315^\circ) \\ &\Downarrow \\ (x, y) &= (\sqrt{2}, -\sqrt{2}) \end{aligned} \right\}$$

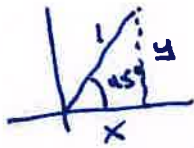


Defn:

$$\begin{aligned} \sin \theta &= y/r \\ \cos \theta &= x/r \end{aligned}$$

$$\begin{aligned} x &= r \cdot \cos \theta \\ y &= r \cdot \sin \theta \end{aligned}$$

1/11



$$\sin(45^\circ) = \frac{y}{1} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \approx 0.707$$

$$\cos(45^\circ) = \frac{x}{1} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \approx 0.707$$



$$x = \frac{1}{2}$$

$$y^2 + x^2 = 1^2$$

$$y^2 + \frac{1}{4} = 1$$

$$y^2 = 3/4$$

$$y = \sqrt{3}/2$$

x=y: $2x^2 = 1$
 $x^2 = 1/2$
 $x = \sqrt{1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = y$

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}$$

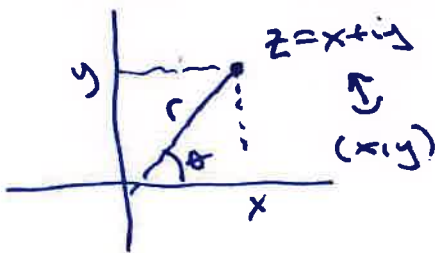
$$\cos(60^\circ) = \frac{1}{2}$$

Complex numbers:

$$\boxed{\begin{matrix} x = r \cdot \cos \theta \\ y = r \sin \theta \end{matrix}}$$

$$z = x + iy = r \cdot \cos \theta + i \cdot r \sin \theta$$

$$\underline{z = r \cdot (\cos \theta + i \sin \theta)}$$



Complex numbers using polar coordinates

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

Note:

i) $|z| = r$

since $|\cos \theta + i \sin \theta|$



$$= 1$$

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

$$= \cos^2 \theta + \sin^2 \theta$$

$$= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$

$$= \frac{x^2 + y^2}{r^2} = 1$$

Could derive this by

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

ii) $\boxed{e^{i\theta} = \cos \theta + i \sin \theta}$

de Moivre's formula / Euler's formula:

$$z = r \cdot (\cos \theta + i \sin \theta) \Rightarrow z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

$$z = r \cdot e^{i\theta} \Rightarrow z^n = r^n \cdot e^{i \cdot n\theta}$$

Ex: $z^5 = 1$

$$z = r \cdot (\cos \theta + i \sin \theta)$$

$$r^5 \cdot (\cos 5\theta + i \sin 5\theta) = 1$$

Modulus:

$$r^5 = 1$$

$$r = 1$$

Angles:

$$\cos 5\theta + i \sin 5\theta = \cos 0^\circ + i \sin 0^\circ$$

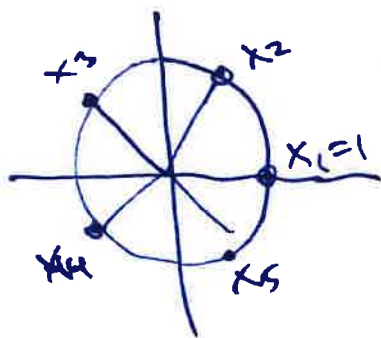
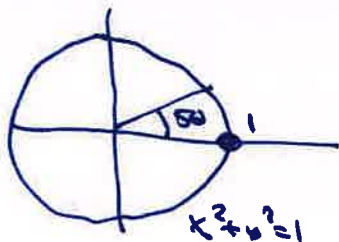
$$\Rightarrow 5\theta = 0^\circ + k \cdot 360^\circ, \quad k \text{ int.}$$

$$\theta = \frac{0^\circ + k \cdot 360^\circ}{5}$$

$$= k \cdot 72^\circ, \quad k = 0, 1, 2, 3, 4,$$

$$\theta = \underline{0^\circ}, \underline{72^\circ}, \underline{144^\circ}, \underline{216^\circ}, \underline{288^\circ}$$

$$\underline{z} = 1 \cdot (\cos \theta + i \sin \theta), \quad \theta = \underline{0^\circ}, \underline{72^\circ}, \underline{144^\circ}, \underline{216^\circ}, \underline{288^\circ}$$



```
1 import numpy as np
2 import math
3 import cmath
4
5 z = 3 - 2j
6 w = complex(3,1)
7
8 z*w
9
0 z/w
1
2 # The modulus of z
3 abs(z)
4
5 # The angle of z, in radians
6 cmath.phase(z)
7
8 # The angle of z, in degrees
9 math.degrees(cmath.phase(z))
0
1 # The polar coordinates of z, with angles in radians
2 cmath.polar(z)
3
4 # Examples of square (real) matrices
5 A = np.array([[0,-1],[1,0]])
6 B = np.random.randn(3,3)
7
8 print(A)
9 print(B)
0
1 # Eigenvalues of the matrices (in Lecture 4 next Friday)
2 np.linalg.eigvals(A)
3 np.linalg.eigvals(B)
4
5
```