Plan

1. Quadratic functions and optimization
2. Minimum variance portfolio

Part 2: at 12-13 today
(Zoom) (not 11-12)

Review:
Kuhn-Tucker problem in std. form:

$$\boxed{\max f(\underline{x}) \text{ when } \begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}}$$

Candidate pts:

i) $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC + C + CSC

$$\boxed{\underset{\text{FOC}}{\mathcal{L}'_{\underline{x}} = 0} \quad \underset{\text{C}}{g_i(\underline{x}) \le a_i} \quad \underset{\text{CSC}}{\lambda_i \ge 0,\ \lambda_i(g_i(\underline{x}) - a_i) = 0}}$$

ii) $\underline{x}^*$ adm. pt. where NDCQ fails

Env. Thm (for Lagrange / Kuhn-Tucker problems)

$$\boxed{\frac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \underline{\lambda}^*(a))}$$

1 Quadratic functions = polynomial fn. of degree 2

Ex: $f(x,y,z) = \underbrace{4x^2 - 4xy + 2y^2 + 4xz + 2z^2}_{\text{quadratic form}} \underbrace{- 4x - 4z}_{\text{linear form}} \underbrace{+ 3}_{\text{constant}}$

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \quad B = (-4 \ \ 0 \ -4) \quad C = 3$$

Result: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

i) $f'(\underline{x}) = \begin{pmatrix} f'_{x_1} \\ \vdots \\ f'_{x_n} \end{pmatrix} = 2A\underline{x} + B^T$

ii) $H(f) = 2A$

Ex: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + c$ $A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ $B = (-4 \; 0 \; 4)$

$c = 3$

$D_1 = 4 \quad D_2 = 4$

$D_3 = 2(-4) + 2 \cdot 4 = 0$

Find max/min $f(\underline{x})$:

i) Stationary pts: $f'(\underline{x}) = \underline{0}$ FOC

$$2A \cdot \underline{x} + B^T = \underline{0}$$

$$2A\underline{x} = -B^T$$

$$A\underline{x} = -\tfrac{1}{2}B^T$$

$$\left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ -2 & 2 & 0 & 0 \\ 2 & 0 & 2 & +2 \end{array}\right) \to \left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\cdot\frac{1}{2}$, $-\frac{1}{2}$, -1

$x = 1 - z$

$4x = 4 - 4z$

$4x = 2 + 2(1-z) - 2z$

$y = 1 - z$

z free

Stat pts:

$(x, y, z) = (1-z, 1-z, z)$

where z is free

ii) Classify: $H(f)(\underline{x}^*) = 2A$ pos. semidef. No concl. from sec. der. test

iii) Convex/concave: $H(f) = 2A$ — " —

$\iff f$ convex $\Rightarrow (1-z, 1-z, z)$ are global min (for all z)

$z = 1$

all global min pts have the same value

Conclusion: $f_{min} = f(0,0,1) = 1$

Ex: max/min $f(\underline{x}) = x^2 + 2xy + 4y^2 + z^2$ when $x+y+z=1$

$= \underline{x}^T A \underline{x}$ $\qquad B\underline{x} = 1$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$L = \underline{x}^T A \underline{x} - \lambda \cdot (B\underline{x} - 1) \longleftarrow$ quadratic function in $\underline{x}$

FOC: $L'(\underline{x}) = \boxed{2A\underline{x} - \lambda(B^T) = \underline{0}}$

C: $\boxed{B\underline{x} = 1}$

$2A\underline{x} = \lambda \cdot B^T$

$A\underline{x} = \frac{\lambda}{2} \cdot B^T$

$\underline{x} = A^{-1}(\frac{\lambda}{2} B^T)$

$= \frac{\lambda}{2}(A^{-1}B^T)$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D_1 = 1$, $D_2 = 3$, $D_3 = 1 \cdot 3 = 3$ pos. defn. (invertible)

$B\underline{x} = 1$

$B \cdot \frac{\lambda}{2} A^{-1}B^T = 1$

$\frac{\lambda}{2} \cdot BA^{-1}B^T = 1$

$\lambda = \dfrac{2}{BA^{-1}B^T}$

$\underline{x} = \dfrac{1}{BA^{-1}B^T} \cdot A^{-1}B^T$

Compute: $A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^T$

$$= \begin{pmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$BA^{-1}B^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2$$

$$\underline{x} = \frac{1}{2} \cdot \frac{1}{3} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \lambda = 1 \longrightarrow$$

Cand pts: $(x, y, z; \lambda) = (1/2, 0, 1/2; 1)$

SOC: $L(\underline{x}; 1) = \underline{x}^T A \underline{x} - (B\underline{x} - 1) = h(\underline{x})$

$H(h) = 2A$ pos. defn $\Rightarrow h$ convex $\Rightarrow f_{min} = f(1/2, 0, 1/2) = 1/4 + 1/4 = \underline{1/2}$

Ex: max/min $f(\underline{x}) = \underline{x}^T A \underline{x}$ when $x^2+y^2+z^2=6$

$= x^2+2xy+4y^2+z^2$

$\underline{x}^T \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{x} = 6$

$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 6)$

$\underline{x}^T \underline{x} = 6$

FOC: $L'(\underline{x}) = 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0}$

C: $\underline{x}^T \cdot \underline{x} = 6$

$2A\underline{x} = 2\lambda I \underline{x}$ $\quad |:2$

$A\underline{x} = \lambda I \underline{x} = \lambda \underline{x}$

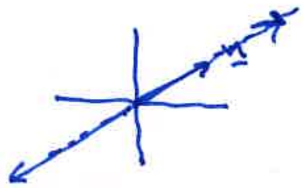
$(A-\lambda I)\underline{x} = 0 \iff |A-\lambda I| = 0$ or $\underline{x} = \underline{0}$

$\underline{x}$ is an eigenvector of A with eigenvalue λ

$\underline{0}^T \cdot \underline{0} = 0 \neq 6$ not possible

$\underline{x}^T \cdot \underline{x} = 6$ $\qquad \|x\|^2 = 6$

$\|x\| = \sqrt{6}$



$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)\cdot[(1-\lambda)(4-\lambda)-1] = 0$

$\lambda = 1$ or $\lambda^2 - 5\lambda + 3 = 0$

$\lambda = \frac{5 \pm \sqrt{25-12}}{2}$

$= \frac{5 \pm \sqrt{13}}{2}$

$\lambda_1 = 1$: $E_1 = \text{span}(\underline{v}_1)$ two candidate pts $f = 6$

$\lambda_2 = \frac{5+\sqrt{13}}{2} \simeq 4.30$ two candidate pts max $\rightarrow f = 3(5+\sqrt{13})$

$\lambda_3 = \frac{5-\sqrt{13}}{2} \simeq 0.70$ two candidate pts min $\rightarrow f = 3(5-\sqrt{13})$

$\underline{x}$ eigenvector of A in E_λ: $f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \lambda \underline{x} = \lambda \underline{x}^T \underline{x} = 6\lambda$

② Minimum variance portfolio

Universe: n securities with returns $R_1, \dots, R_n$ with known

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

vector of expected returns — covariance matrix

$\mu_i = E(R_i)$ $\qquad \sigma_i^2 = Var(R_i)$, $\quad \sigma_{ij} = Cov(R_i, R_j)$

Portfolio: $\underline{w} = (w_1, w_2, \dots, w_n) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

w_i: weight of security #i in the portfolio

$$\boxed{w_1 + w_2 + \dots + w_n = 1} \iff \underline{e}^T \cdot \underline{w} = 1$$

$R = w_1R_1 + \dots + w_nR_n$: (return of the portfolio)

$$E(R) = w_1 \cdot E(R_1) + \dots + w_n \cdot E(R_n) = (\mu_1 \; \mu_2 \cdots \mu_n) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \underline{\mu}^T \cdot \underline{w}$$

$$Var(R) = Cov(R, R) = Cov(w_1R_1 + \dots + w_nR_n, \; w_1R_1 + \dots + w_nR_n)$$
$$= w_1^2 \cdot \sigma_1^2 + w_1w_2 \cdot \sigma_{12} + \dots + w_n^2\sigma_n^2$$
$$= \underline{w}^T \Sigma \underline{w} \iff \sigma_R = \sqrt{Var(R)}$$

Note:

a) Σ is symmetric and positive semidefinite

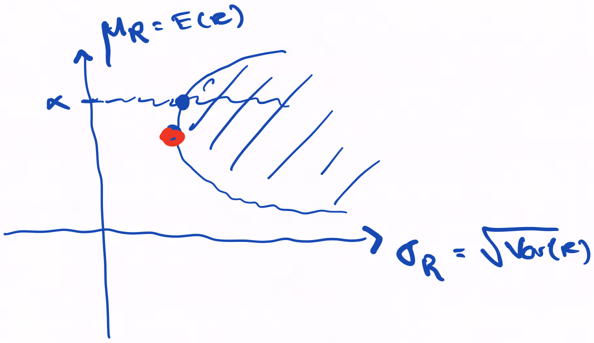
b) Reasonable assumptions:

i) Σ positive definite. $\quad \rightarrow$ If not, $\underline{w}^T\Sigma\underline{w} = 0$ for some $\underline{w}^* \neq \underline{0}$

ii) $\{\underline{\mu}, \underline{e}\}$ lin. independent $\left(\underline{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right)$ $\rightarrow$ If not, $\underline{\mu} = c \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix}$

Problem A: $\min \underbrace{\underline{w}^T \Sigma \underline{w}}_{= \mathrm{Var}(R)}$ when $\underline{e}^T \cdot \underline{w} = 1$

Problem B: $\min \underbrace{\underline{w}^T \Sigma \underline{w}}_{= \mathrm{Var}(R)}$ when $\begin{cases} \underline{e}^T \cdot \underline{w} = 1 \\ \underbrace{\underline{\mu}^T \cdot \underline{w}}_{E(R)} = \alpha \end{cases}$ for a given α



(A) $\mathcal{L} = \underline{w}^T \Sigma \underline{w} - \lambda(\underline{e}^T \underline{w} - 1)$

FOC: $\mathcal{L}'(\underline{w}) = 2\Sigma \underline{w} - \lambda \cdot \underline{e} = \underline{0}$

C: $\underline{e}^T \cdot \underline{w} = 1$

$2\Sigma \underline{w} = \lambda \underline{e}$ — Σ pos. defn. $\rightarrow |\Sigma| \neq 0$

$\Sigma \underline{w} = \frac{\lambda}{2} \underline{e}$

$\underline{w} = \Sigma^{-1}\left(\frac{\lambda}{2}\underline{e}\right) = \frac{\lambda}{2} \Sigma^{-1} \underline{e}$

$\underline{e}^T \left(\frac{\lambda}{2} \Sigma^{-1} \underline{e}\right) = 1$

$\frac{\lambda}{2} \underline{e}^T \Sigma^{-1} \underline{e} = 1$

$\lambda^* = \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}}$

Cand. pt:

$\underline{w}^* = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \Sigma^{-1} \underline{e}$; $\lambda = \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}}$

min. variance portfolio

SOC: $h = \mathcal{L}(\underline{w}; \lambda^*) = \underline{w}^T \Sigma \underline{w} - \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot (\underline{e}^T \underline{w} - 1)$

$H(h) = 2\Sigma$ pos. defn $\Rightarrow$ h convex $\rightarrow$ $\underline{w}^*$ is min

(B) $\mathcal{L} = \underline{w}^T \Sigma \underline{w} - \lambda_1(\underline{e}^T \underline{w} - 1) - \lambda_2(\underline{\mu}^T \underline{w} - \alpha)$

FOC: $\mathcal{L}'(\underline{w}) = 2\Sigma \underline{w} - \lambda_1 \underline{e} - \lambda_2 \underline{\mu} = \underline{0}$

C: $\begin{cases} \underline{e}^T \cdot \underline{w} = 1 \\ \underline{\mu}^T \underline{w} = \alpha \end{cases}$

$$2\Sigma \underline{w} = \lambda_1 \underline{e} + \lambda_2 \underline{\mu}$$
$$\underline{w} = \Sigma^{-1}\left(\tfrac{\lambda_1}{2}\underline{e} + \tfrac{\lambda_2}{2}\underline{\mu}\right) = \tfrac{\lambda_1}{2}\Sigma^{-1}\underline{e} + \tfrac{\lambda_2}{2}\Sigma^{-1}\underline{\mu}$$

$$\underline{e}^T\underline{w} = \tfrac{\lambda_1}{2}\underline{e}^T\Sigma^{-1}\underline{e} + \tfrac{\lambda_2}{2}\underline{e}^T\Sigma^{-1}\underline{\mu} = 1$$
$$\underline{\mu}^T\underline{w} = \tfrac{\lambda_1}{2}\underline{\mu}^T\Sigma^{-1}\underline{e} + \tfrac{\lambda_2}{2}\underline{\mu}^T\Sigma^{-1}\underline{\mu} = \alpha$$

$$\Updownarrow$$

$$(\underline{e}^T\Sigma^{-1}\underline{e})\cdot\lambda_1 + (\underline{e}^T\Sigma^{-1}\underline{\mu})\lambda_2 = 2$$
$$(\underline{\mu}^T\Sigma^{-1}\underline{e})\cdot\lambda_1 + (\underline{\mu}^T\Sigma^{-1}\underline{\mu})\lambda_2 = 2\alpha$$

<u>Note:</u> $\begin{pmatrix} \underline{e}^T\Sigma^{-1}\underline{e} & \underline{e}^T\Sigma^{-1}\underline{\mu} \\ \underline{\mu}^T\Sigma^{-1}\underline{e} & \underline{\mu}^T\Sigma^{-1}\underline{\mu} \end{pmatrix} = \begin{pmatrix} \underline{e}^T \\ \underline{\mu}^T \end{pmatrix} \Sigma^{-1} (\underline{e} \mid \underline{\mu})$ <u>pos. defn.</u>

$$\Rightarrow \boxed{\begin{aligned} \begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} &= \begin{pmatrix} \underline{e}^T\Sigma^{-1}\underline{e} & \underline{e}^T\Sigma^{-1}\underline{\mu} \\ \underline{\mu}^T\Sigma^{-1}\underline{e} & \underline{\mu}^T\Sigma^{-1}\underline{\mu} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 2\alpha \end{pmatrix} \\ \underline{w} &= \tfrac{\lambda_1}{2}\Sigma^{-1}\underline{e} + \tfrac{\lambda_2}{2}\Sigma^{-1}\underline{\mu} \end{aligned}}$$

One candidate pt. $\xRightarrow{\text{SOC}}$ minimum

<u>Explanation:</u> Why is $\underbrace{\begin{pmatrix} \underline{e}^T \\ \underline{\mu}^T \end{pmatrix} \Sigma^{-1} (\underline{e} \mid \underline{\mu})}_{B}$ pos. defn.

<u>Note:</u>

B is a 2×2 symmetric matrix

$\Rightarrow$ look at its quadratic fn

$$q(x,y) = (x\ y)\, B \begin{pmatrix} x \\ y \end{pmatrix} = (x\underline{e}^T + y\underline{\mu}^T)\,\Sigma^{-1}(x\underline{e} + y\underline{\mu})$$

i) $q(x,y) \geq 0$ since Σ^{-1} is pos. defn. symmetric matrix

Σ^{-1} is pos. defn. symmetric:

* $\Sigma^{-1} = \frac{1}{|\Sigma|} \text{adj}(\Sigma)$ is symmetric since Σ is symmetric
* Σ pos defn $\Rightarrow$ all eigenvalues of Σ are positive $(\lambda_i > 0)$ $(\lambda_1, \lambda_2, \dots, \lambda_n)$

$\Rightarrow$ eigenvalues of Σ^{-1} are $1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n > 0$

$\Rightarrow$ Σ^{-1} pos. defn.

$q(x,y) = \underline{v}^T \Sigma^{-1} \underline{v} \geq 0$ with $\underline{v} = x\underline{e} + y\underline{\mu}$

(since Σ^{-1} pos. defn.)

ii) $q(x,y) = 0 \Rightarrow \underline{v}^T \Sigma^{-1} \underline{v} = 0$ with $\underline{v} = x\underline{e} + y\underline{\mu}$

$\Rightarrow \underline{v} = \underline{0}$ (since Σ^{-1} is pos. defn.)

$\Rightarrow x\underline{e} + y\underline{\mu} = \underline{0}$

$\Rightarrow x = y = 0$ (since $\{\underline{e}, \underline{\mu}\}$ are lin. independent)

Since $q(x,y) \geq 0$, and $q(x,y) = 0 \Rightarrow (x,y) = (0,0)$, it follows that q is <u>positive definite</u>.

$\Rightarrow$ Symmetric matrix of q = $B = \begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix}$ pos. defn.

Plan

1 More examples: Constrained optimization

$$\max f(x,y) = \ln(x^2 y) - x - y \quad \text{when} \begin{cases} x+y \geq 4 & 1\cdot(-1) \\ x \geq 1 & 1\cdot(-1) \\ y \geq 1 & 1\cdot(-1) \end{cases} \quad \text{KT}$$

std.

$$\max f = \ln(x^2 y) - x - y \quad \text{when} \begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases} \quad \text{KT, std}$$

$$\mathcal{L} = \ln(x^2 y) - x - y - \lambda_1(-x-y+4) - \lambda_2(-x+1) - \lambda_3(-y+1)$$

$$= 2\ln(x) + \ln(y) - x - y + \lambda_1(x+y-4) + \lambda_2(x-1) + \lambda_3(y-1)$$

FOC: $\mathcal{L}'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0$

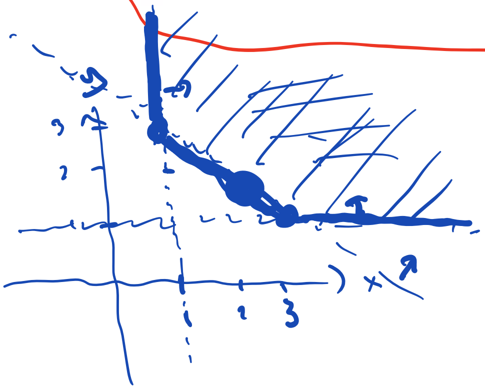
$\mathcal{L}'_y = \frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0$

C: $x+y \geq 4$, $x \geq 1$, $y \geq 1$

CSC: $\lambda_1 \geq 0$, $\lambda_1(x+y-4) = 0$

$\lambda_2 \geq 0$, $\lambda_2(x-1) = 0$

$\lambda_3 \geq 0$, $\lambda_3(y-1) = 0$



Assume $x=1$: $1 + \lambda_1 + \lambda_2 = 0$ impossible

$\Rightarrow$ we must have: $x > 1, \lambda_2 = 0$

Assume $y=1$: $0 + \lambda_1 + \lambda_3 = 0$

$\Rightarrow \lambda_1 = 0, \lambda_3 = 0$

$\frac{2}{x} - 1 + 0 + 0 = 0 \Rightarrow x = 2$

$x+y = 3$ impossible

$\Rightarrow$ we must have: $y > 1, \lambda_3 = 0$

Assume $x+y=4$:

$\frac{2}{x} - 1 + \lambda_1 = 0$ $\quad \lambda_1 = 1 - \frac{2}{x} = 1 - \frac{1}{y}$

$\frac{1}{y} - 1 + \lambda_1 = 0$ $\quad \frac{2}{x} = \frac{1}{y}$

$2y = x$

$x + y = 2y + y = 4$, $3y = 4$ $\quad y = 4/3$, $x = 8/3$

Kandidate pt: $(x,y;\lambda_1,\lambda_2,\lambda_3) = (8/3, 4/3; 1/4, 0, 0)$ ok

SOC: $h = \mathcal{L}(x,y;\ 1/4, 0, 0)$

$= 2\ln x + \ln y - x - y + \frac{1}{4}(x+y-4)$

$H(h) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$ neg. defn. $\Rightarrow$ h concave

$\Rightarrow$ $(8/3, 4/3)$ is max

$f_{max} = f(8/3, 4/3) =$

$= \ln\left((8/3)^2(4/3)\right) - \frac{8}{3} - \frac{4}{3}$

$= \ln(256/27) - 4$