Plan

1. Kuhn-Tucker problems
2. Second order conditions
3. Non-degenerate constraint qualification

Review: Lagrange problem max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Necessary conditions: $\mathcal{L} = f(\underline{x}) - \lambda_1(g_1(\underline{x}) - a_1) - \dots - \lambda_m(g_m(\underline{x}) - a_m)$

i) Lagrange conditions $\boxed{\text{FOC}+\text{C}}$: $\mathcal{L}'_{x_i} = 0 \quad g_i(\underline{x}) = a_i$

ii) Adm. pts where NDCQ fails

SOC: $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC+C, and $\mathcal{L}(\underline{x}; \underline{\lambda}^*)$ convex $\Rightarrow$ $\underline{x}^*$ min

— " — concave $\Rightarrow$ $\underline{x}^*$ max

$\boxed{\text{EVT: } f \text{ cont. on compact } D \Rightarrow f \text{ attains max/min on } D}$

① **Kuhn-Tucker problems** = constrained optimization problem with closed inequality ($\leq, \geq$) constraints

Defn: A Kuhn-Tucker problem in standard form: max $f(\underline{x})$ subject to $\begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

Ex: max $f(x,y,z) = x^2+y^2+z^2$ when $xyz \geq 1 \quad | \cdot (-1)$

$-xyz \leq -1$ ← std. form

min $f(x,y,z) = x^2+y^2+z^2$ when $x+2y-z \leq 6$

$\Updownarrow$

max $-f(x,y,z) = -x^2-y^2-z^2$

Ex: max $f(x,y,z) = x^2y^2z^2$ when $x^2+y^2+z^2+x^2y^2z^2 \le 4$

std. form

$$\mathcal{L} = x^2y^2z^2 - \lambda(x^2+y^2+z^2+x^2y^2z^2-4)$$

FOC:

$$\mathcal{L}'_x = 2xy^2z^2 - \lambda(2x + 2xy^2z^2) = 0$$

$$\mathcal{L}'_y = x^2\cdot 2y\cdot z^2 - \lambda(2y + x^2\cdot 2y\cdot z^2) = 0$$

$$\mathcal{L}'_z = x^2y^2\cdot 2z - \lambda\cdot(2z + x^2y^2\cdot 2z) = 0$$

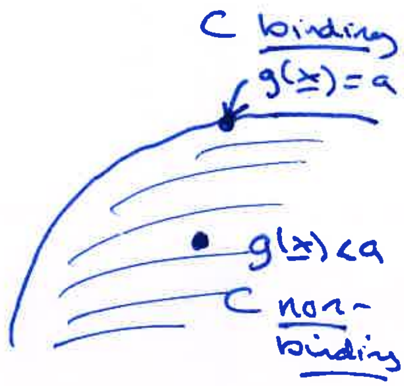
C: $x^2+y^2+z^2+x^2y^2z^2 \le 4$

CSC: $\lambda \ge 0$

$$\lambda\cdot(x^2+y^2+z^2+x^2y^2z^2-4) = 0$$

Complementary slackness conditions

Kuhn-Tucker conditions: FOC + C + CSC



$\lambda \ge 0$ and $\lambda(g(\underline{x}) - a) = 0$

$\Updownarrow$

$$\boxed{\begin{array}{l} \lambda = 0 \text{ if } g(\underline{x}) < a \\ \lambda \ge 0 \text{ if } g(\underline{x}) = a \end{array}}$$

ND

C: $x^2+y^2+z^2+x^2y^2z^2 < 4$

CSC: $\lambda = 0$

FOC: $2xy^2z^2 = 0$, $2x^2yz^2 = 0$, $2x^2y^2z = 0$

$\Downarrow$

$x=0$ or $y=0$ or $z=0$

Many pts, all with $f=0$

B

$x^2+y^2+z^2+x^2y^2z^2 = 4$

$\lambda \ge 0$

FOC:

$$2x\left(y^2z^2 - \lambda(1+y^2z^2)\right) = 0$$

$$2y\left(x^2z^2 - \lambda(1+x^2z^2)\right) = 0$$

$$2z\left(x^2y^2 - \lambda(1+x^2y^2)\right) = 0$$

$x=0$ or $\lambda = \dfrac{y^2z^2}{1+y^2z^2}$

$y=0$ or $\lambda = \dfrac{x^2z^2}{1+x^2z^2}$

$z=0$ or $\lambda = \dfrac{x^2y^2}{1+x^2y^2}$

$y^2z^2 = \lambda\cdot(1+y^2z^2)$

$\lambda = \dfrac{y^2z^2}{1+y^2z^2}$

i) $x=0$ or $y=0$ or $z=0$: all pts have $f=0$

ii) $\lambda = \dfrac{y^2z^2}{1+y^2z^2} = \dfrac{x^2z^2}{1+x^2z^2} = \dfrac{x^2y^2}{1+x^2y^2}$

$$y^2z^2(1+x^2z^2) = (1+y^2z^2)x^2z^2$$

$$y^2z^2 + \cancel{x^2y^2z^4} = x^2z^2 + \cancel{x^2y^2z^4}$$

$$z^2(y^2-x^2) = 0 \Rightarrow y^2 = x^2$$

$\to x^2z^2(1+\cancel{x^2y^2}) = x^2y^2(1+\cancel{x^2z^2})$

$x^2z^2 = x^2y^2$

$z^2 = y^2$

$$\Rightarrow x^2 = y^2 = z^2 \qquad \lambda = \frac{y^2 z^2}{1+y^2 z^2} = \frac{x^2 \cdot x^2}{1+x^2 \cdot x^2}$$

C: $x^2+y^2+z^2+x^2y^2z^2 = 4$

$3x^2 + x^6 = 4$

$x^6 + 3x^2 - 4 = 0$

$u = x^2$: $u^3 + 3u - 4 = 0$ ← $u=1$ is a solution

$(u-1)(u^2+u+4) = 0$

$u = 1$ or $u^2+u+4=0$

$x^2 = 1$ $\qquad u = \frac{-1 \pm \sqrt{1-4\cdot 4}}{2}$

no solutions

Candidate pts: $x^2 = y^2 = z^2 = 1$, $\lambda = 1/2$

$\Rightarrow (\pm 1, \pm 1, \pm 1; 1/2)$ $\quad f = 1$

Sol. of FOC+C+CSC

EVT: f cont., D closed and bounded $\Rightarrow$ there is a max
✓yes ✓ yes ?
($\leq, \geq$)

C: $x^2+y^2+z^2+x^2y^2z^2 \leq 4$

D is bounded $\Rightarrow$ there is a max

$x^2 \leq 4$: $-2 \leq x \leq 2$

$y^2 \leq 4$: $-2 \leq y \leq 2$

$z^2 \leq 4$: $-2 \leq z \leq 2$

$f(\pm 1, \pm 1, \pm 1) = 1$ is best candidate for max among ordinary candidate pts (FOC+C+CSC)

NDCQ: Are there any adm. pts where NDCQ fails?

② NDCQ in the Kuhn-Tucker case

C: $g(\underline{x}) \le a$

$x^2+y^2+z^2+x^2y^2z^2 \le 4$

B: $x^2+y^2+z^2+x^2y^2z^2 = 4$: ~~NDCQ:~~

NB:

NDCQ (B):

$\text{rk}\begin{pmatrix} g'_x & g'_y & g'_z \end{pmatrix} = 1$ $\quad \text{rk}\begin{pmatrix} 2x+2xy^2z^2 & 2y+2yx^2z^2 & 2z+2zx^2y^2 \end{pmatrix} = 1$

NDCQ fails: $\quad 2x(1+y^2z^2)=0 \quad 2y(1+x^2z^2)=0 \quad 2z(1+x^2y^2)=0$

$\quad x=0 \quad y=0 \quad z=0$

$(0,0,0)$ is not adm. with $x^2+y^2+z^2+x^2y^2z^2=4$

NDCQ (NB):

no condition

Concl: In the example, there are no adm. pts where NDCQ fails.

⇓

Concl for the problem:

$f_{max} = 1$ at $(x,y,z) = (\pm 1, \pm 1, \pm 1)$ with $\lambda = 1/2$

NDCQ for a general Kuhn-Tucker case:

$$\left.\begin{array}{l} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{array}\right\}$$

Let $\underline{x}^*$ be an adm pt (satisfying all constraints) and let J be the matrix obtained from

$$\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \vdots & & & \vdots \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

by keeping rows that correspond to binding constraints.

NDCQ: rk J is maximal

Ex: NDCQ, KT problem with several constraints

$$\max\ -2x^2-y^2-3z^2 \quad \text{when} \quad \begin{cases} -x+y-2z \le -3 \\ -x-y \le -3 \end{cases}$$

C: $-x+y-2z \le -3$
$-x-y \le -3$

full Jacobian:

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

Cases:

i) B-B: $-x+y-2z=-3$, $-x-y=-3$ $\quad \mathrm{rk}\begin{pmatrix} \boxed{\begin{matrix} -1 & 1 \\ -1 & -1 \end{matrix}} & \begin{matrix} -2 \\ 0 \end{matrix} \end{pmatrix} = 2$

$M = 1-(-1) = 2 \neq 0$ ok

ii) B-NB: $-x+y-2z=-3$, $-x-y<-3$ $\quad \mathrm{rk}\begin{pmatrix} -1 & 1 & -2 \end{pmatrix} = 1$ ok

iii) NB-B: $-x+y-2z<-3$, $-x-y=-3$ $\quad \mathrm{rk}\begin{pmatrix} -1 & -1 & 0 \end{pmatrix} = 1$ ok

iv) NB-NB: $-x+y-2z<-3$, $-x-y<-3$ $\quad$ no condition ok

Conclusion: No adm. pts where NDCQ fails.

In general: $\max f(\underline{x})$ when
$$\begin{cases} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases} \leftarrow \text{std. form}$$

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a_2) - \ldots - \lambda_m (g_m(\underline{x}) - a_m)$$

FOC: $\mathcal{L}'_{x_1} = 0$, $\mathcal{L}'_{x_2} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) \le a_1$, $g_2(\underline{x}) \le a_2$, $\vdots$, $g_m(\underline{x}) \le a_m$

CSC: $\lambda_1 \ge 0$, $\lambda_1(g_1(\underline{x}) - a_1) = 0$; $\lambda_2 \ge 0$, $\lambda_2(g_2(\underline{x}) - a_2) = 0$; $\vdots$; $\lambda_m \ge 0$, $\lambda_m(g_m(\underline{x}) - a_m) = 0$

Thm (Necessary condition for Kuhn–Tucker problems in std form)
If $\underline{x}^*$ is a maximum pt in a Kuhn–Tucker problem in std. form, and NDCQ is satisfied at $\underline{x}^*$, then there are Lagrange multipliers $\underline{\lambda}^*$ such that $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC + C + CSC.

(3) SOC: (in the Kuhn-Tucker case)

Thm:
If $(\underline{x}^*, \underline{\lambda}^*)$ satisfy FOC + C + CSC in a Kuhn-Tucker problem in std. form, and

$$h(\underline{x}) = \mathcal{L}(\underline{x}; \underline{\lambda}^*) \text{ is concave}$$

then $\underline{x}^*$ is maximum point.

Ex: $\max\ x^2y^2z^2$ u.c. $x^2+y^2+z^2+x^2y^2z^2 \leq 4$

Candidate: $(\pm 1, \pm 1, \pm 1; 1/2)$ ← satisfies FOC + C + CSC

$$h(x,y,z) = \mathcal{L}(x,y,z;1/2) = x^2y^2z^2 - \frac{1}{2}(x^2+y^2+z^2+x^2y^2z^2-4)$$

$$= -\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + \frac{1}{2}x^2y^2z^2 + 2$$

$$H(h) = \begin{pmatrix} -1+y^2z^2 & 0 & 0 \\ 0 & -1+x^2z^2 & 0 \\ 0 & 0 & -1+x^2y^2 \end{pmatrix} = \begin{pmatrix} y^2z^2-1 & 0 & 0 \\ 0 & x^2z^2-1 & 0 \\ 0 & 0 & x^2y^2-1 \end{pmatrix}$$

$D_1 = y^2z^2 - 1$ can be both pos./neg.

h is not concave.

No conclusion from SOC.

Plan

1 Envelope theorem for constrained optimization problems

Ex: $\max f(x,y,z,w) = xw - yz$ wh $\begin{cases} x^2+y^2=16 \\ 4z^2+9w^2=36 \end{cases}$

Lagrange problem (from Lecture 7, Part I)

Concl: $f_{max} = 12$ at $(x,y,z,w) = (0,4,-3,0)$, $(0,-4,3,0)$ with $\lambda_1 = 3/8$, $\lambda_2 = 1/6$

What if: i) $4z^2+9w^2 = 35$ — What happen with the maximum value?

ii) $f = xw - 2yz$

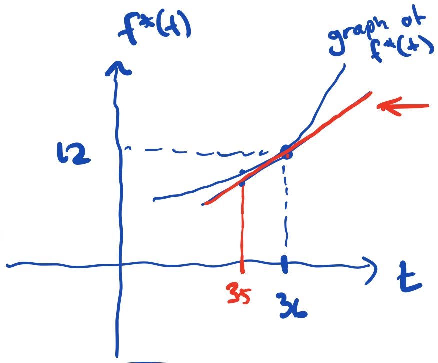
i) Consider: $\max f = xw - yz$ wh $\begin{cases} x^2+y^2=16 \\ 4z^2+9w^2=t \end{cases}$ ← t = parameter

Know: $f^*(t)$ max. value fn.

$\left.\begin{array}{l} x^*(t) \\ y^*(t) \\ z^*(t) \\ w^*(t) \end{array}\right\}$ max pt.

$\left.\begin{array}{l} \lambda_1^*(t) \\ \lambda_2^*(t) \end{array}\right\}$ Lagrange mult. at max pt.

$f^*(36) = 12$
$x^*(36) = 0$
$y^*(36) = 4$ or -4
$z^*(36) = -3$ or 3
$w^*(36) = 0$
$\lambda_1^*(36) = 3/8$
$\lambda_2^*(36) = 1/6$



slope of tangent line $= \lambda_2^*(36) = 1/6$

Envelope Thm: $\dfrac{df^*(t)}{dt} = \mathcal{L}'_t(\underline{x}^*(t); \underline{\lambda}^*(t))$

We want to compute the right hand side

Estimate max. value when $t = 35$:

$f^*(35) \simeq f^*(36) + \Delta t \cdot \frac{df^*(t)}{dt}(36)$

$= 12 + (-1) \cdot 1/6$

$= 12 - 1/6 \simeq 11.83$

$\mathcal{L} = xw - yz - \lambda_1(x^2+y^2-16) - \lambda_2(4z^2+9w^2-t)$

$= \ldots + \lambda_2 t$

$\mathcal{L}'_t = \lambda_2 \Rightarrow \mathcal{L}'_t(\underline{x}^*(t); \underline{\lambda}^*(t)) = \lambda_2^*(t)$

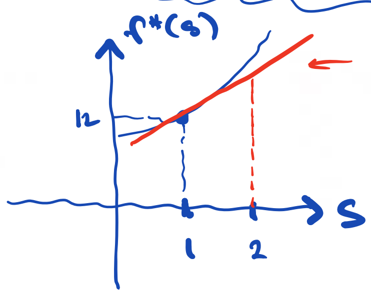
At $t = 36$: $\dfrac{df^*(t)}{dt} = \lambda_2^*(36) = 1/6$

ii) $\max\ f = xw - s \cdot yz$ when $\begin{cases} x^2+y^2=16 \\ 4z^2+9w^2=36 \end{cases}$ s = parameter

$s=1$: $f^*(1) = 12$

$x^*(1)=0 \quad y^*(1)=4 \text{ or } -4 \quad z^*(1)=-3 \text{ or } 3 \quad w^*(1)=0$

$\lambda_1^*(1) = 3/8 \quad \lambda_2^*(1) = 1/6$



slope of tangent line = 12

Using envelope thm:

$$\mathcal{L} = xw - syz - \lambda_1(x^2+y^2-16) - \lambda_2(4z^2+9w^2-36)$$

$$\frac{df^*(s)}{ds} = \mathcal{L}'_s(\underline{x}^*(s); \underline{\lambda}^*(s))$$

$$\mathcal{L}'_s = -yz \Rightarrow \frac{df^*(s)}{ds} = -y^*(s)z^*(s)$$

$s=1$: $= -y^*(1)z^*(1)$

$= -4\cdot(-3) = 12$

$-(-4)\cdot 3 = 12$

Estimate: $f^*(2) \simeq f^*(1) + \overset{(2-1)}{\Delta s} \cdot \dfrac{df^*(s)}{ds}$

$= 12 + 1 \cdot 12$

$= 24$

Note: $\lambda_i = \dfrac{df^*(a_i)}{da_i}$

"marginal change in max/min value when you change a_i, the constant in the i'th constraint"

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

general Lagrange problem