Plan

1. Lagrange problems
2. Second order conditions

Midterm GRA6035: 15-16 on paper
Midterm ELE3781: Home exam in the next weeks.

Lecture 7 Part 2: Monday 12-13. (Zoom invite will come)

Review: Unconstrained optimization

max/min $f(\mathbf{x})$

Method: (i) Find all stationary pts

$f'_{x_1} = \dots = f'_{x_n} = 0$ (FOC)

(ii) Classify each stationary pt $\mathbf{x}^*$

$H(f)(\mathbf{x}^*)$ pos. defn $\Rightarrow \mathbf{x}^*$ local min
" neg. defn $\Rightarrow$ — " — max
" indefn. $\Rightarrow \mathbf{x}^*$ saddle pt

Second derivative test

(iii) Determine if any local max/min are global max/min

$H(f)(\mathbf{x})$ pos. semi-defn. for all $\mathbf{x}$ $\Rightarrow$ f convex
" neg. — " — $\Rightarrow$ f concave

f convex $\Rightarrow$ stationary pts are global min
f concave $\Rightarrow$ — " — global max

① **Lagrange problems** = Constrained optimization problem with equality constraints.

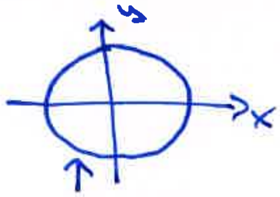
General form: max/min $f(\underline{x})$ subject to $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

$f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ — objective function

Constraints — admissible pts

$D \subseteq \mathbb{R}^n$ subset of $\mathbb{R}^n$ consisting of all admissible pts, i.e. all pts satisfying all constraints

Ex: min $f(x,y) = xy$ when $x^2 + y^2 = 10$



$D = \{(x,y) : x^2 + y^2 = 10\}$
circle, $r = \sqrt{10}$, center $(0,0)$.

Lagrange's method = method of Lagrange multipliers:

$L = L(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m)$ ← Lagrangian

$= f(x_1, \ldots, x_n) - \lambda_1 \cdot [g_1(x_1, \ldots, x_n) - a_1] - \lambda_2 [g_2(x_1, \ldots, x_n) - a_2] - \ldots$

$\ldots - \lambda_m [g_m(x_1, \ldots, x_n) - a_m]$

$= f(\underline{x}) - \lambda_1 (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a_2) - \ldots - \lambda_m (g_m(\underline{x}) - a_m)$

Lagrange conditions: = conditions for stationary points of L

FOC $\begin{cases} L'_{x_1} = f'_{x_1} - \lambda_1 \cdot (g_1)'_{x_1} - \ldots - \lambda_m \cdot (g_m)'_{x_1} = 0 \\ L'_{x_2} = f'_{x_2} - \lambda_1 \cdot (g_1)'_{x_2} - \ldots - \lambda_m (g_m)'_{x_2} = 0 \\ \vdots \\ L'_{x_n} = f'_{x_n} - \lambda_1 (g_1)'_{x_n} - \ldots - \lambda_m (g_m)'_{x_n} = 0 \end{cases}$

C $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Candidate points = all solutions of FOC + C ← $m+n$ equations in $m+n$ variables

Note:

$$L'_{\lambda_1} = -(g_1(\underline{x})-a_1)=0 \iff g_1(x)=a_1$$

$$\vdots \qquad\qquad\qquad \vdots$$

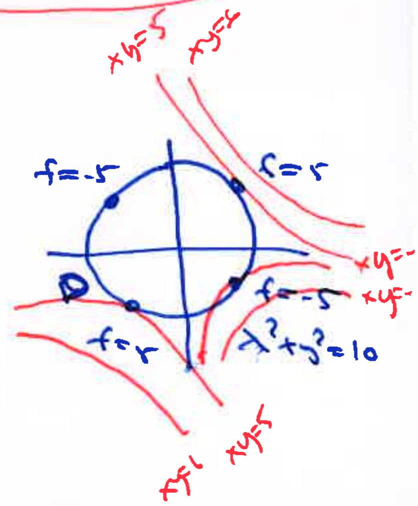
$$L'_{\lambda_m} = -(g_m(\underline{x})-a_m)=0 \iff g_m(\underline{x})=a_m$$

Thm: Necessary conditions for Lagrange problems

If $(x_1^*,x_2^*,\dots,x_n^*)=\underline{x}^*$ is a max/min in a Lagrange problem, and the non-degenerate constraint qualification (NDCQ) holds at $\underline{x}^*$, then there are Lagrange multipliers $(\lambda_1^*,\dots,\lambda_m^*)=\underline{\lambda}^*$ such that $(\underline{x}^*;\underline{\lambda}^*)$ satisfies the Lagrange conditions (FOC+C).

$\underline{x}^*$ max/min $\underset{\text{NDCQ}}{\Longrightarrow}$ $(\underline{x}^*,\underline{\lambda}^*)$ solves FOC+C for some $\underline{\lambda}^*$

Ex: min $f=xy$ when $\underbrace{x^2+y^2}_{g(x,y)}=\underbrace{10}_{a}$



$$L = xy - \lambda\cdot(x^2+y^2-10)$$

FOC $\begin{cases} L'_x = y-\lambda\cdot 2x = 0 \\ L'_y = x-\lambda\cdot 2y = 0 \end{cases}$ C: $x^2+y^2=10$

(1) $y = 2\lambda x$

(2) $x-2\lambda(2\lambda x)=0$

$x-4\lambda^2 x=0$

$x(1-4\lambda^2)=0$ $\to$ $x=0$ or $\lambda^2=1/4$

$x=0$	$\lambda=1/2$	$\lambda=-1/2$
$y=0$	$y=x$	$y=-x$
$0^2+0^2\neq 10$	$x^2+x^2=10$	$x^2+x^2=10$
impossible	$x^2=5$	$x^2=5$
	$x=\pm\sqrt5=y$	$x=\pm\sqrt5$, $y=-x$
	$(\sqrt5,\sqrt5;1/2)$	$(\sqrt5,-\sqrt5;-1/2)$
	$(-\sqrt5,-\sqrt5;1/2)$	$(-\sqrt5,\sqrt5;-1/2)$

Candidate pts:

$(\sqrt5,\sqrt5;1/2)$ $f=5$ $\qquad$ $(\sqrt5,-\sqrt5;-1/2)$ $f=-5$

$(-\sqrt5,-\sqrt5;1/2)$ $f=5$ $\qquad$ $(-\sqrt5,\sqrt5;-1/2)$ $f=-5$

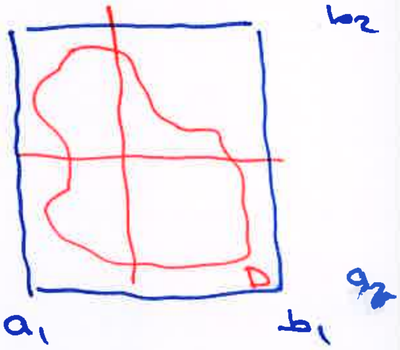
Defn: A subset D of $\mathbb{R}^n$ is compact if it is closed and bounded.

A set defined by $=, \leq, \geq$ is closed.
OK for Lagrange problems

Defn: A subset D of $\mathbb{R}^n$ is bounded (bounded) if there are (fixed, finite) numbers $a_1, \dots, a_n$, $b_1, \dots, b_n$ such that

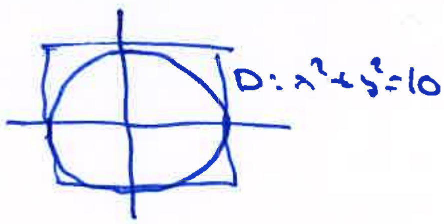
$$a_1 \leq x_1 \leq b_1$$
$$a_2 \leq x_2 \leq b_2$$
$$\vdots$$
$$a_n \leq x_n \leq b_n$$

for all $(x_1, \dots, x_n)$ in D.



Ex: $x^2+y^2=10$ is bounded

$$-\sqrt{10} \leq x \leq \sqrt{10}$$
$$-\sqrt{10} \leq y \leq \sqrt{10}$$



Extreme Value Theorem (EVT)

If f is a continuous function defined on a compact subset D of $\mathbb{R}^n$, then f attains a maximum and a minimum on D.

In Lagrange problems, we consider the objective function f and the set $D = \{(x_1, \dots, x_n) : g_1(\mathbf{x}) = a_1, g_2(\mathbf{x}) = a_2, \dots, g_m(\mathbf{x}) = a_m\}$ of adm pts. D is always closed, but we must check if it is bounded.

Ex: max/min $f(x,y,z,w) = xw - yz$ when $\begin{cases} x^2+y^2=16 \\ 4z^2+9w^2=36 \end{cases}$

$$L = xw - yz - \lambda_1(x^2+y^2-16) - \lambda_2(4z^2+9w^2-36)$$

FOC:
$L'_x = w - \lambda_1 \cdot 2x = 0$
$L'_y = -z - \lambda_1 \cdot 2y = 0$
$L'_z = -y - \lambda_2 \cdot 8z = 0$
$L'_w = x - \lambda_2 \cdot 18w = 0$

C: $x^2+y^2=16$, $4z^2+9w^2=36$

Note: f cont., D closed, bounded } EVT ⇒ There is a max and min.

$D = \{(x,y,z,w) : x^2+y^2=16,\ 4z^2+9w^2=36\}$

D is bounded since
$-4 \le x \le 4$
$-4 \le y \le 4$
$-3 \le z \le 3$
$-2 \le w \le 2$

Solve FOC+C:

(1)+(4): $w = 2\lambda_1 x$
$x - 18\lambda_2 \cdot (2\lambda_1 x) = 0$
$x - 36\lambda_1\lambda_2 x = 0$
$x(1 - 36\lambda_1\lambda_2) = 0$
$x=0$ or $\lambda_1\lambda_2 = 1/36$

(2)+(3): $z = -2\lambda_1 y$
$-y - 8\lambda_2 \cdot (-2\lambda_1 y) = 0$
$-y(1 - 16\lambda_1\lambda_2) = 0$
$y=0$ or $\lambda_1\lambda_2 = 1/16$

(a) $x=0, y=0$: $w=0, z=0$
$0^2+0^2 \neq 16$ impossible

(b) $x=0, \lambda_1\lambda_2 = 1/16$: $w=0$
$x^2+y^2=16$: $y = \pm 4$
$4z^2+9w^2=36$: $z = \pm 3$
$\lambda_1 = -\frac{1}{2}\frac{z}{y}$ $\quad \lambda_1\lambda_2 = 1/16$

(c) $\lambda_1\lambda_2 = 1/36, y=0$: $z=0$
$x^2+y^2=16$: $x = \pm 4$
$4z^2+9w^2=36$: $w = \pm 2$
$\lambda_1 = \frac{1}{2}\frac{w}{x}$ $\quad \lambda_1\lambda_2 = 1/36$

(b) $(x,y,z,w) = (0,4,3,0)$ $\lambda_1 = -3/8$ $\lambda_2 = -\frac{1}{3\cdot 2} = -\frac{1}{6}$ $f=-12$

$(0,4,-3,0)$ $\lambda_1 = 3/8$ $\lambda_2 = 1/6$ $f=12$

$(0,-4,3,0)$ $\lambda_1 = 3/8$ $\lambda_2 = 1/6$ $f=12$

$(0,-4,-3,0)$ $\lambda_1 = -3/8$ $\lambda_2 = -1/6$ $f=-12$

(c) $(x,y,z,w) = (4,0,0,2)$ $\lambda_1 = 1/4$ $\lambda_2 = 1/4$ $f=8$

$(4,0,0,-2)$ $\lambda_1 = -1/4$ $\lambda_2 = -1/4$ $f=-8$

$(-4,0,0,2)$ $\lambda_1 = -1/4$ $\lambda_2 = -1/4$ $f=-8$

$(-4,0,0,-2)$ $\lambda_1 = 1/4$ $\lambda_2 = 1/4$ $f=8$

Conclusions:

i) EVT: there is max/min

ii) FOC+C: Best candidates for max: $(0,4,-3,0; 3/8, 1/6)$ $f=12$

$(0,-4,3,0; 3/8, 1/6)$ $f=12$

Best candidates for min: $(0,4,3,0; -3/8, -1/6)$ $f=-12$

$(0,-4,-3,0; -3/8, -1/6)$ $f=-12$

Assuming NDCQ holds: $f_{max} = 12$ $f_{min} = -12$

② Second order conditions

Set of adm pts.

$D = \{(x,y,z) : x+2y+3z=6\}$

Ex: min $f(x,y,z) = x^2+y^2+z^2$ when $x+2y+3z=6$

$L = x^2+y^2+z^2 - \lambda(x+2y+3z-6)$

FOC:
$$\begin{cases} L'_x = 2x-\lambda\cdot 1 = 0 \\ L'_y: 2y-\lambda\cdot 2 = 0 \\ L'_z = 2z-\lambda\cdot 3 = 0 \end{cases} \quad \begin{matrix} x = \lambda/2 \\ y = \lambda \\ z = 3\lambda/2 \end{matrix} \longrightarrow$$

C: $x+2y+3z=6$

$\lambda/2 + 2\lambda + 3\cdot 3\lambda/2 = 6 \quad |\cdot 2$

$\lambda + 4\lambda + 9\lambda = 12$

$14\lambda = 12$

$\lambda = 12/14 = 6/7$

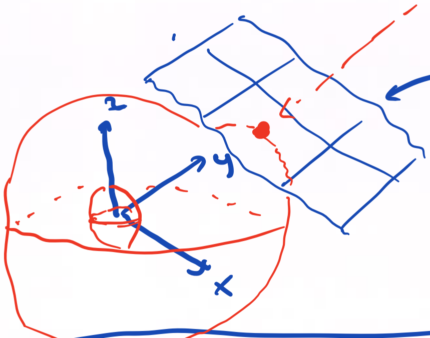
$x = 3/7$

$y = 6/7$

$z = \frac{3}{2}\cdot\frac{6}{7} = \frac{9}{7}$

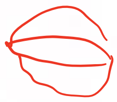
Solve FOC+C $\longrightarrow$ Candidate pts $(3/7, 6/7, 9/7; 6/7)$

D bounded $\overset{EVT}{\Longrightarrow}$ there is a min ↑ cannot use EVT.



D: $x+2y+3z=6$
a plane in $\mathbb{R}^3$
not bounded

$f = x^2+y^2+z^2 = c$ is a sphere (ball)



Second order condition (SOC) for Lagrange problems

If $(\underline{x}^*; \underline{\lambda}^*)$ is a point that satisfies FOC+C in a Lagrange problem, then we have:

$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$ convex $\Longrightarrow$ $\underline{x}^*$ is a min

— " — concave $\Longrightarrow$ $\underline{x}^*$ is a max

Ex: Candidate pt $(3/7, 6/7, 9/7; 6/7)$

SOC: $h(x,y,z) = L(x,y,z; 6/7) = x^2+y^2+z^2 - 6/7\cdot(x+2y+3z-6)$

$$H(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ pos. defn at all pts $\Rightarrow$ h convex

⇓ (SOC)

$(x,y,z) = (3/7, 6/7, 9/7)$ is a min pt

Plan

1. Non-degenerate constraint qualification (NDCQ)
2. Envelope theorem

① NDCQ (non-deg. constr. qualification)

C: $g_1(x_1,\dots,x_n)=a_1$
$\vdots$
$g_m(x_1,\dots,x_n)=a_m$

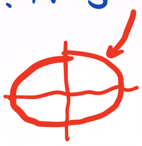
$$J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \dots & \partial g_1/\partial x_n \\ \vdots & & & \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \dots & \partial g_m/\partial x_n \end{pmatrix}$$

(Jacobian) $m \times n$ matrix

NDCQ: $\operatorname{rk} J = m$

Ex: min $f(x,y)=xy$ when $x^2+y^2=10$

C: $x^2+y^2=10$



$J = (2x \quad 2y)$

NDCQ: $\operatorname{rk} J = 1$

Assume that NDCQ fails (i.e. $\operatorname{rk} J < 1$):
$2x = 2y = 0$
$\Rightarrow (x,y)=(0,0)$, not adm.

Note: If there are any adm pts where NDCQ fails, then these pts could be max/min in the Lagrange problem.

FOC + C ⟶ Ordinary candidate pts.
Adm pts. (C) where NDCQ fails ⟶ Exceptional candidate pts

Ex: max $f(x,y,z)=z$ when $x^2+y^2+z^3=0$

$\mathcal{L} = z - \lambda(x^2+y^2+z^3)$

FOC + C:
$$\begin{cases} \mathcal{L}'_x = -\lambda \cdot 2x = 0 \\ \mathcal{L}'_y = -\lambda \cdot 2y = 0 \\ \mathcal{L}'_z = 1 - \lambda \cdot 3z^2 = 0 \\ x^2+y^2+z^3=0 \end{cases}$$

$\lambda = 0$ or $x=0$

~~$1=0$~~ imp. ~~$y=0$, $1=0$~~ impossible

~~$z^3=0$~~ $z=0$

$\Rightarrow$ no pts that satisfy FOC + C

NDCQ: $x^2+y^2+z^3=0$

$\text{rk}(2x \;\; 2y \;\; 3z^2) = 1$ ← NDCQ

NDCQ fails: $2x=2y=3z^2=0$

$\Rightarrow (0,0,0)$, adm.

Exceptional candidate pt: $(x,y,z)=(0,0,0)$

$f(0,0,0)=0$

Note: $x^2+y^2+z^3=0$

$z^3=-(x^2+y^2)\leq 0$

$z \leq 0$ ⟶ **is max**

Methods for Lagrange problems:

Alt. A: D bounded $\overset{\text{EVT}}{\Rightarrow}$ there is a max/min

i) Find all ordinary candidate pts: Solve FOC + C

ii) Find exceptional cand. pts: Adm pts where NDCQ fails

⇓

List of all cand pts: Highest f-value ⇒ max

Lowest " ⇒ min

Alt B: Find a cand. pt $(\underline{x}^*; \lambda^*)$ satisfying FOC+C

Check this point with SOC

$h(\underline{x}) = \mathcal{L}(\underline{x};\lambda^*)$ convex $\Rightarrow \underline{x}^*$ is min

— " — concave $\Rightarrow \underline{x}^*$ is max

② Envelope Theorem: Will be covered in Lecture 8.