Plan

1 Functions in several variables
2 Unconstrained optimization
3 Convex and concave functions

Midterm exam: 08/10/21
Curriculum: Lecture 1–6

Plenary Session 2:
Thu 07/10/2 at 18–20.45
Problems: Problemset 5–6

Review:

i) Definiteness of a quadratic form $f(\underline{x}) = \underline{x}^T A \underline{x}$ in n variables

- using eigenvalues $\lambda_1, \dots, \lambda_n$ of A
- using principal minors $D_1, \dots, D_n$ and $\Delta_1, \dots, \Delta_n$ of A

(a) $D_1, D_2, \dots, D_n > 0 \iff A$ pos. defn.
$D_1 < 0, D_2 > 0, \dots, (-1)^n D_n > 0 \iff A$ neg. defn.

(b) $\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff A$ pos semidefn.
$\Delta_1 \leq 0, \Delta_2 \geq 0, \dots, (-1)^n \Delta_n \geq 0 \iff A$ neg. semidefn.

(c) RRC: if $\operatorname{rk} A = r < n$, ~~then~~ we have:
$D_1, D_2, \dots, D_r > 0 \Rightarrow A$ pos. semidefn.
$D_1 < 0, D_2 > 0, \dots, (-1)^r D_r > 0 \Rightarrow A$ neg. semidefn

ii) Markov chains: A regular $\iff A^m > 0$ for some $m \geq 1$

A regular Markov chain $\Rightarrow$ Equilibrium state = unique state vector in E_1

iii) Orthogonal diagonalization: A symmetric $\Updownarrow$ $P^T A P = D$ for an orthogonal matrix P (i.e. such that $P^T = P^{-1}$)

$P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ where $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ is an orthonormal set of eigenvectors:

$$\left\{ \begin{array}{l} \underline{v_1} \cdot \underline{v_1} = 1 \quad \underline{v_2} \cdot \underline{v_2} = 1 \quad \dots \quad \underline{v_n} \cdot \underline{v_n} = 1 \\ \underline{v_i} \cdot \underline{v_j} = 0 \quad \text{when } i \neq j \end{array} \right\}$$

① Functions in several variables

Unconstrained optimization problem: max/min $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$

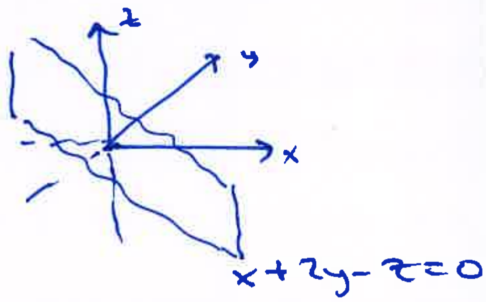
Ex: 1) $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$

2) $f(x,y) = x^2y^3+y^2-2y$

3) $f(x,y,z) = \ln(x+2y-z)$

Defn: The domain of definition D_f of a fn. $f(\underline{x})$ in n variables is the subset of $\mathbb{R}^n$ where the function is defined.

Ex: 1), 2) $D_f = \mathbb{R}^n$ 3) D_f: $x+2y-z>0$



The graph of f is the set of points $(x_1, x_2, \ldots, x_n, y)$ where $(x_1, \ldots, x_n)$ is in D_f and $y = f(x_1, \ldots, x_n)$

The ~~range~~ range V_f of f is the set of all possible function values of f.

Defn: The point $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a maximum (global maximum) for f if $f(\underline{x}) \le f(\underline{x}^*)$ for all $\underline{x}$ in D_f. The value $f(\underline{x}^*)$ is called the maximum value of f.

Similar for minimum = global minimum. $f(\underline{x}^*) \ge f(\underline{x})$ for all $\underline{x}$ in D_f

Defn: f is continuous at $\underline{x}^*$ if $\underline{x}^*$ is in D_f and
$$\lim_{\underline{x} \to \underline{x}^*} f(\underline{x}) = f(\underline{x}^*)$$

f is continuous if it is continuous at all pts in D_f.

All "usual" functions are continuous.

A subset D of $\mathbb{R}^n$ is closed if all boundary pts of D are included in D, and open if no boundary pts of D are included in D.

Ex: $D = \{(x,y) : x^2 + y^2 < 1\}$

Typically: Sets defined by $<, >, \neq$ are open
Sets defined by $\leq, \geq, =$ are closed

$D = \mathbb{R}^n$: open and closed

The general method for solving the unconstrained optimization problem works when f is "nice". That is: f is a C^2 function defined on an open subset D

" all second order partial derivatives are continous functions

Ex: $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$
$D = D_f = \mathbb{R}^4$ open
f is a polynomial $\Rightarrow$ f is C^2

Result: If f is C^2, then $H(f)$ is symmetric. That is $f''_{x_i x_j} = f''_{x_j x_i}$.

2 Unconstrained optimization

Method for solving max/min $f(\underline{x})$

1. Find all stationary pts of f; i.e. points where

$$f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0 \quad \text{(FOC)}$$

Candidate pts = stationary pts

2. Classify all stationary pts; i.e. as local max, local min, or saddle point.

Defn: $\underline{x}^*$ local max : $f(\underline{x}) \le f(\underline{x}^*)$ for points $\underline{x}$ close to $\underline{x}^*$

$\underline{x}^*$ local min : $f(\underline{x}) \ge f(\underline{x}^*)$ — " —

$\underline{x}^*$ saddle pt : all other cases

Second derivative test:

Let $\underline{x}^*$ be a stationary pt of f. We consider

Symm. matrices

$$H(f)(\underline{x}^*) = \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_n} \\ f''_{x_2x_1} & f''_{x_2x_2} & \dots & f''_{x_2x_n} \\ \vdots & & & \\ f''_{x_nx_1} & f''_{x_nx_2} & \dots & f''_{x_nx_n} \end{pmatrix}(\underline{x}^*)$$

(SOC)

Then we have:

$H(f)(\underline{x}^*)$ positive defn. $\Rightarrow$ $\underline{x}^*$ local min for f

— " — negative defn. $\Rightarrow$ $\underline{x}^*$ local max — " —

— " — indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt.

Note: If $H(f)(\underline{x}^*)$ is pos/neg semidefn. but not pos/neg defn then this test is inconclusive. $\rightarrow$ Must use another method.

3. Determine if any of the local max/min are global max/min.

Ex: $f(x,y,z) = x^4 + y^4 + z^4$

$$\left.\begin{aligned} f'_x &= 4x^3 = 0 \\ f'_y &= 4y^3 = 0 \\ f'_z &= 4z^3 = 0 \end{aligned}\right\}$$ Stat. pts: $(x,y,z) = (0,0,0)$

$$H(f) = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix} \Rightarrow H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pos. semidefn but not pos. defn.

neg. semidefn. but not neg. defn.

$\Downarrow$

Cannot use second derivative test to classify $(0,0,0)$

Alt. Method: Use defn.

$f(0,0,0) = 0$

$f(x,y,z) = x^4 + y^4 + z^4 \geq 0$ when (x,y,z) close to $(0,0,0)$

$\Rightarrow (0,0,0)$ is local min

(In fact, $H(f)(x,y,z)$ pos. semidefn. for all (x,y,z), hence f is convex, and $(0,0,0)$ is global min.)

Ex: $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$

① Stationary pts:

$f'_x = 2x+y = 0$
$f'_y = 2y+x+z = 0$
$f'_z = 2z+y+w = 0$
$f'_w = 2w+z = 0$

$$\underbrace{\begin{aligned} 2x+y &= 0 \\ x+2y+z &= 0 \\ y+2z+w &= 0 \\ z+2w &= 0 \end{aligned}}$$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 3/4$
$D_3 = 1\cdot 3/4 - \frac{1}{2}\cdot\frac{1}{2} = 1/2$
$D_4 = 1\cdot 1/2 - 1/2\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\right) = 5/16 > 0$

$2A\cdot \underline{x} = \underline{0}$

$A\cdot \underline{x} = \underline{0}$

$\Downarrow$

$\underline{x} = \underline{0}$

Stationary pts:
$\underline{x} = \underline{0}$
$(x,y,z,w) = (0,0,0,0)$

② Classification:

$$H(f)(0,0,0,0) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}(0,0,0,0)$$

$$= 2A$$

$D_1 = 2 \quad D_2 = 3 \quad D_3 = 4 \quad D_4 = 5 \Rightarrow H(f)(\underline{0})$ pos. defn.

$\Downarrow$

$\underline{x}^* = \underline{0}$ is local min

In general, A and 2A has the same determinants
matrix is multiplied by 2 $\Rightarrow$
D_1 is multiplied by 2
D_2 is — " — $4 = 2^2$
D_3 is — " — $8 = 2^3$
D_4 is — " — $16 = 2^4$

③ Is $\underline{0}$ is global min?

Since f is pos. defn. quadr. form, $\underline{x}^* = \underline{0}$ is a global min.

$f(\underline{0}) = 0$
$f(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$

③ Convex / concave functions.

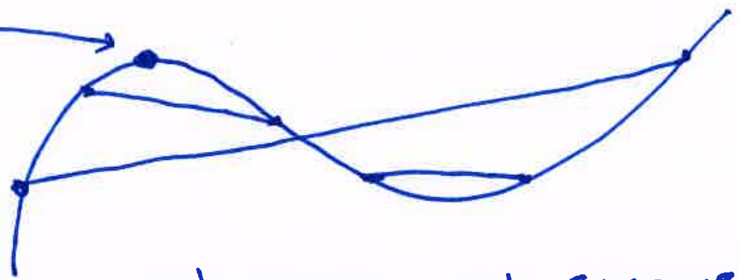
Defn: A function f is convex / concave if the following condition holds:

For any two points A and B ~~[illegible]~~ on the graph of f, the line segment $[A, B]$ lies over/under the graph of f (or on the graph of f).

convex

concave

local max but not global max



not convex, not concave

Convex / concave optimization:

If f is convex, then any stationary point is a global min.
If f is concave, then any stationary point is a global max.

How to determine if f is convex / concave:

Criterion: (SOC)

If $H(f)(\underline{x})$ is positive semidefin. for all $\underline{x}$ in D_f, then f is convex.

If $H(f)(\underline{x})$ is negative semidefin. for all $\underline{x}$ in D_f, then f is concave.

Ex: $f(x,y,z) = x^3 + y^3 + z^3 - xyz$

$f'_x = 3x^2 - yz$
$f'_y = 3y^2 - xz$
$f'_z = 3z^2 - xy$

$$H(f) = \begin{pmatrix} \boxed{6x} & -z & -y \\ -z & 6y & -x \\ -y & -x & 6z \end{pmatrix}$$

$D_1 = 6x \leftarrow$ can be both pos. and neg.
$D_2 =$
$D_3 =$

f is neither convex nor concave

Note:

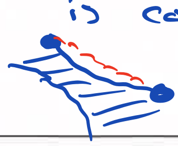
f is convex $\Leftrightarrow$ $H(f)(\underline{x})$ pos. semidefin for all $\underline{x}$ in D_f

f is concave $\Leftrightarrow$ " neg. semidefin. — " —

Defn: A subset D of $\mathbb{R}^n$ is called convex if the following condition holds:

If A, B are in D, then the line segment $[A, B]$ is contained in D.

 convex

 not convex

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

$f'_x = 2xy^3 = 0 \Rightarrow x=0$ or $y=0$

$f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$ | $2y-2=0$, $y=1$ | $0+0-2=0$ impossible

Stat. pts: $(x,y) = (0,1)$

$$H(f) = \begin{pmatrix} \boxed{2y^3} & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix} \Rightarrow H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$D_1 = 2$, $D_2 = 4$ } pos. def.

$\Downarrow$

$(0,1)$ is local min

$f(0,1) = -1$

f is neither convex nor concave

$f(x,y) = x^2y^3 + y^2 - 2y$

$f(1,y) = y^3 + y^2 - 2y \to -\infty$ when $y \to -\infty$

$f(1,-10) = -1000 + 100 + 20 = -880 < -1$

no global min

Plan

1 Quadratic form: $f(\underline{x}) = \underline{x}^T A \underline{x}$

max/min $f(\underline{x})$

Result: 1 $(2A)\underline{x} = \underline{0}$ is the first order conditions

$|A| \neq 0$: $\underline{x} = \underline{0}$ unique stationary pt

$|A| = 0$: inf. many stationary pts, including $\underline{x} = \underline{0}$

2 $H(t) = 2A$ – constant matrix
– same definiteness as A

$\Downarrow$

Conclusions:

1) A pos. defn: $\underline{x} = \underline{0}$ is unique global min
2) A pos. semidefn: $\underline{x} = \underline{0}$ is global min
3) A neg defn.: $\underline{x} = \underline{0}$ is unique global max
4) A neg semidefn. $\underline{x} = \underline{0}$ is global max
5) A indefinite: $\underline{x} = \underline{0}$ is saddle point

Ex: $f(x,y,z) = e^{1-x^2-y^2+xy-z^2}$, $D_f = \mathbb{R}^3$

max/min f

$= e^{1+u}$, where $u = \underbrace{-x^2-y^2+xy-z^2}_{\text{quadratic form}}$

Inner fn: $u(x,y,z) = -x^2-y^2+xy-z^2$

$$A = \begin{pmatrix} -1 & 1/2 & 0 \\ 1/2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$D_1 = -1$
$D_2 = 3/4$
$D_3 = -1 \cdot D_2 = -3/4$

neg. defn.

$\Leftarrow$ u has a unique global max $u(0,0,0) = 0$

$\boxed{u \le 0}$:

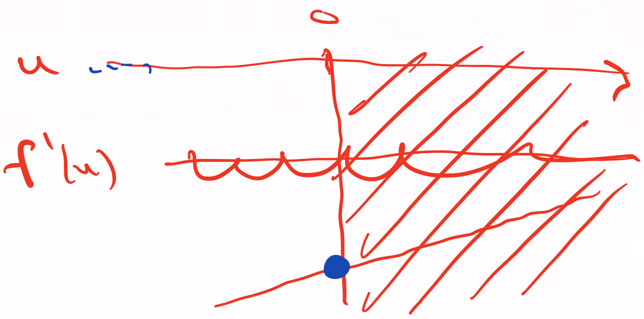
$V_u = (\leftarrow, 0]$

Outer fn: $f(u) = e^{1+u}$

$f'(u) = e^{1+u} \cdot 1 = e^{1+u} > 0$

$f_{max} = e^{1+0} = e^1 = e$

at $u=0$, i.e. $(x,y,z) = (0,0,0)$



f_{min}: $\lim_{u \to -\infty} e^{1+u} = 0$

$V_f = (0, e]$

Max for f: $f_{max} = f(0,0,0) = e$

No min.

(2) Envelope theorem

Ex: $\max \; 1 + 2x - x^2 = f(x)$

$$f'(x) = 2 - 2x = 0$$
$$x = 1$$

$$f''(x) = -2 < 0 \Rightarrow f \text{ concave}$$

$$\Rightarrow f_{max} = f(1) = 2$$

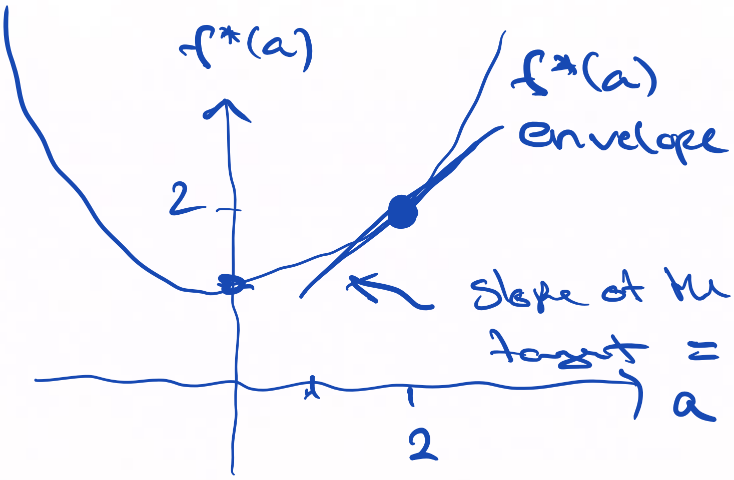
Max-problem with parameter:

$$\max \; f(x;a) = 1 + ax - x^2$$

We know: $\boxed{\begin{array}{l} x^*(2) = 1 \\ f^*(2) = 2 \end{array}}$

max point when $a = 2$

max value when $a = 2$



graph of $f^*(a)$, max value function

$f' = a - 2x = 0$

$x = \underline{a/2}$

$f'' = -2 < 0$

f concave

$x^*(a) = \underline{\underline{a/2}}$ is global max

$f^*(a) = f(a/2)$

$= 1 + a \cdot a/2 - (a/2)^2$

$= \underline{\underline{1 + a^2/4}}$

$$\boxed{\text{Envelope thm: } \quad \frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))}$$

Slope of tangent at $f^*(a)$

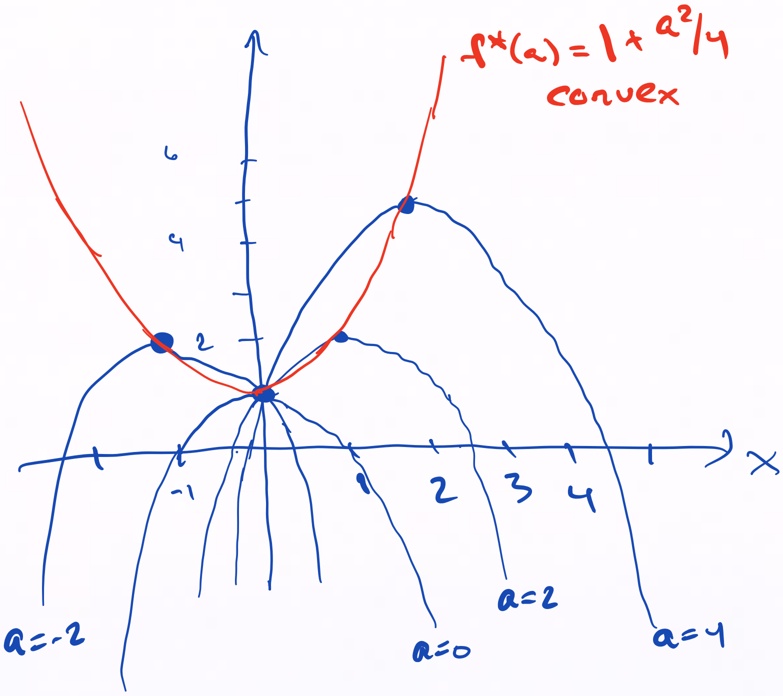
$f = 1 + ax - x^2$

$f'_a = x$

$f'_a(x^*(a))$

$= x^*(a) = \underline{\underline{a/2}}$

See picture on next page.



Blue curves:

Graph of $y = f(x; a)$ for different values of a

In each case, a concave function with maximum marked.

Red curve: The envelope, going through all maximum points

$$\boxed{\frac{d f^*(a)}{da} = f_a'(x^*(a))}$$

$\frac{d f^*(a)}{da}$ $=$ slope of tangent line of max value fn. $f^*(a)$

How to compute the slope of the tangent line

Envelope thm.