Plan

1 Eigenvalues and eigenvectors
2 Diagonalization

Review:

- Know how to compute determinants/minors, matrix ~~multiplication~~ multiplication, (inverses, transpose)
- A : $m \times n$ matrix
  $\left\{ \begin{array}{l} \text{all } r\text{-minors} \\ \text{are zero} \end{array} \right\} \iff \text{rk}\, A < r$
- A : $n \times n$ matrix
  $\boxed{|A| \neq 0 \iff \text{rk}(A) = n \iff \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} \text{ linearly independent (col. vectors of } A)}$

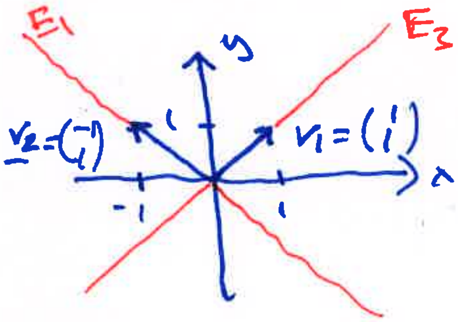
Monday 20/09:
- no TA session
- Plenary Session 1 : A1-040 at 1700-1745
  I will go through selected problems from Lecture 1-4 (Key Problems, [E] Problems, Exam Problems)
  → See list in the lecture plan

① Eigenvalues and eigenvectors

A: $n \times n$ matrix

$$T(\underline{v}) = \underbrace{A \cdot \underline{v}}_{\mathbb{R}^n \text{ (n-vector)}}, \qquad \underline{v} \in \mathbb{R}^n \text{ (n-vector)}$$

$$T: \mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto A \cdot \underline{v}$$



$$\boxed{\begin{aligned} E_3 &: \ T(\underline{v}) = 3\underline{v} \\ E_1 &: \ T(\underline{v}) = 1 \cdot \underline{v} = \underline{v} \end{aligned}}$$

Defn: An eigenvalue of A is a number λ such that $\boxed{A \cdot \underline{v} = \lambda \cdot \underline{v}}$ has non-trivial solution $\underline{v} \neq \underline{0}$. When λ is an eigenvalue of A, the eigenvectors of A with eigenvalue λ are all solutions $\underline{v}$ of $A\underline{v} = \lambda \underline{v}$.
$E_\lambda = \{\underline{v} : A\underline{v} = \lambda\underline{v}\}$ is called the eigenspace of λ.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\quad T(\underline{v}) = A \cdot \underline{v}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A \cdot \underline{v}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \underbrace{3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\lambda \cdot \underline{v}} \leftarrow$$

$\lambda = 3$ is an eigenvalue, $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with $\lambda = 3$.

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \neq \lambda \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

not eigenvector

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \leftarrow$$

$\lambda = 1$ is eigenvalue, $\underline{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvect.

Remark: $A \cdot \underline{v} = \lambda \cdot \underline{v}$

$\Updownarrow$

$A\underline{v} - \lambda \underline{v} = \underline{0}$

$A\underline{v} - \lambda I \underline{v} = 0$

$(A - \lambda I)\underline{v} = \underline{0}$

homogeneous linear system

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix} - \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} (2-\lambda)x + y &= 0 \\ x + (2-\lambda)y &= 0 \end{aligned}$$

$$\underline{\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Method for finding eigenvalues:

λ is an eigenvalue of A $\Leftrightarrow$ $\underbrace{|A - \lambda I| = 0}_{\text{characteristic eqn. of A}}$

Ex:

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

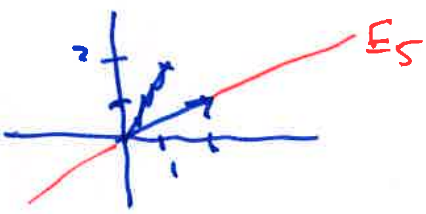
$$= (4-\lambda)(1-\lambda) - 2 \cdot 2 = 0$$

$$= \underbrace{\lambda^2 - 5\lambda = 0}_{\text{char. eqn.}} \qquad \underline{\underline{\lambda = 0}} \text{ or } \underline{\underline{\lambda = 5}}$$

Eigenvectors:

$\lambda = 5$: E_5

$$\underbrace{\begin{pmatrix} 4-5 & 2 \\ 2 & 1-5 \end{pmatrix}}_{A - \lambda I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}$$



$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \qquad -x + 2y = 0, \; y \text{ free} = t$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = \underline{\underline{t \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda = 0}$: E_0

$$\underbrace{\begin{pmatrix} 4-0 & 2 \\ 2 & 1-0 \end{pmatrix}}_{A-\lambda I} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} 4x+2y &= 0 \\ 2x+y &= 0 \end{aligned}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \Big] {-\tfrac{1}{2}} \to \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \qquad 4x+2y=0,\ y \text{ free} = t,\ x = -y/2$$

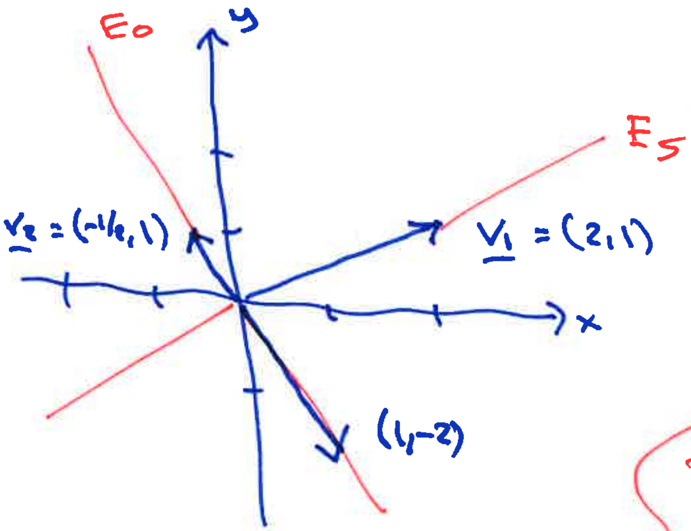
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y/2 \\ y \end{pmatrix} = t \cdot \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$4x+2y=0$

x free

$\downarrow$

$y = -2x$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -2x \end{pmatrix} = x \cdot \underline{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

Both $\underline{v} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can be used as base of E_0

Method for finding eigenvectors with given eigenvalue λ:

Solve $(A - \lambda I) \cdot \underline{v} = \underline{0}$ for the given value of λ, for example using Gauss.

$E_\lambda = \text{Null}(A - \lambda I)$

$\dim E_\lambda = \dim \text{Null}(A - \lambda I)$
$= \#$ free variables in $(A - \lambda I) \cdot \underline{v} = \underline{0}$.

Characteristic equation: $|A-\lambda I|=0$

The case $n=2$:

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0$$

$$(a-\lambda)(d-\lambda)-bc=0$$

$$\lambda^2-a\lambda-d\lambda+ad-bc=0$$

$$\lambda^2-(a+d)\lambda+(ad-bc)=0$$

polynomial eqn. of degree $n=2$.

$$\boxed{\lambda^2-\operatorname{tr}(A)\lambda+\det(A)=0}$$

The case $n\geq 2$:

$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)+\ldots=0$$

Polynomial eqn of degree n. $\longrightarrow$ $\boxed{(-\lambda)^n+\text{lower degree terms}=0}$

Facts:

i) If A is symmetric ($A^T=A$), then A has n eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$
$n\times n$ matrix

ii) If A has n eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$, then we have

$$\boxed{\begin{aligned} &\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\ldots\cdot\lambda_n=\det(A) \\ &\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n=\operatorname{tr}(A)=a_{11}+a_{22}+\ldots+a_{nn} \end{aligned}}$$

Ex:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$+\underline{(2-\lambda)} \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\underline{(2-\lambda)\cdot(\lambda^2 - 6\lambda + 8)} = 0$$

$\underline{\lambda = 2}$ or $\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-2)(\lambda-4) = 0$

$\underline{\lambda = 2}, \ \underline{\lambda = 4}$

$$-\underbrace{(\lambda-2)(\lambda-2)}_{\lambda=2 \text{ has mult. } 2}\underbrace{(\lambda-4)}_{\lambda=4 \text{ mult. } 1} = 0$$

We have three eigenvalues, counted with multiplicity:

$\lambda_1 = \lambda_2 = 2, \ \lambda_3 = 4$

$\underline{\lambda = 2}$: E_2 $(m=2)$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix}$$

$x + z = 0$

y, z free

$$\begin{pmatrix} -z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\dim E_2 = 2}$

$\underline{\lambda = 4}$: E_4 $(m=1)$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix}$$

$-x + z = 0$

$-2y = 0$

z free

$$\begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\dim E_4 = 1}$

$\dim \text{Null}(A - \lambda I) = \#$ free variables

Facts:

i) When λ has mult m, then $1 \leq \dim E_\lambda \leq m$

ii) If A is symmetric, then $\dim E_\lambda = m$ for any eigenvalue λ of multiplicity m.

② Diagonalizable matrices

A
$n \times n$ matrix

Defn: A is diagonalizable if there is an invertible matrix P such that

$$P^{-1}AP = D$$

is a diagonal matrix.

Result: A is diagonalizable if and only if

i) A has n eigenvalues (counted with multiplicity) $\lambda_1, \lambda_2, \dots, \lambda_n$

ii) there are n linearly independent eigenvectors of A $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$

for any eigenvalue of multiplicity m, we have

$$\dim E_\lambda = m$$
$$\dim \text{Null}(A-\lambda I)$$
$$\#\text{free variables}$$

Moreover, in this case, we choose

$$P = \left(\underline{v}_1 \,|\, \underline{v}_2 \,|\, \dots \,|\, \underline{v}_n \right) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

Note:

i) A symmetric $\Rightarrow$ A diagonalizable

ii) If A has n eigenvalues of mult. 1, then A is diagonalizable

Ex: $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ symmetric

Eigenvalues and eigenvectors:

* $\underbrace{\lambda_1 = \lambda_2 = 2}_{\text{mult } 2}, \underbrace{\lambda_3 = 4}_{\text{mult } 1}$

* $E_2: \; z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

dim $E_2 = 2 = m_2$ ok

$E_4: \; z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

dim $E_4 = 1 = m_4$ ok

i) ok: 3 eigenvalue counted with multiplicity

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

ii) ok

$P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$P^{-1}AP = D \quad | P \cdot$

$\cancel{PP^{-1}}AP = PD \quad | \cdot P^{-1}$

$A\cancel{PP^{-1}} = PDP^{-1}$

$A = PDP^{-1}$

$\Longleftrightarrow$

Conclusion:

$P^{-1}AP = D$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Explanation: When $P = (\underbrace{\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n}_{\text{eigenvectors}})$, $D = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{\text{Eigenvalues}}$

then:

$$AP = A(\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) = (A\underline{v}_1 | A\underline{v}_2 | \ldots | A\underline{v}_n)$$

$$= (\lambda_1 \underline{v}_1 | \lambda_2 \underline{v}_2 | \ldots | \lambda_n \underline{v}_n)$$

$$PD = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix} = (\lambda_1 \underline{v}_1 | \lambda_2 \underline{v}_2 | \ldots | \lambda_n \underline{v}_n)$$

$$AP = PD \Rightarrow P^{-1}AP = P^{-1}PD = D \Rightarrow \underline{P^{-1}AP = D}$$

Plan

1. Powers of matrices and eigenvalues
2. Markov chains

① Powers of matrices and eigenvalues

We want to compute A^m $(m >> 1)$

Note: If A is diagonalizable, then $A = PDP^{-1}$

$P^{-1}AP = D$

$$A^m = (PDP^{-1})^m = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\dots(\cancel{P}DP^{-1})$$

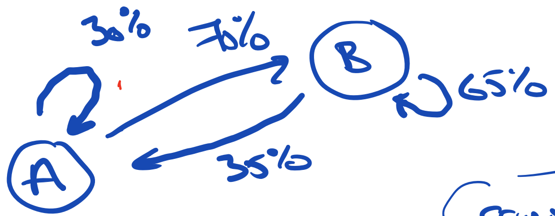
$$= PD^mP^{-1}$$

$$= (v_1|v_2|\dots|v_n)\cdot\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}^m \cdot P^{-1}$$

$$= (v_1|v_2|\dots|v_n)\cdot\begin{pmatrix}\lambda_1^m & & & 0\\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m\end{pmatrix}(v_1|v_2|\dots|v_n)^{-1}$$

② Markov chains and an example

Company A and B share a market



Initial market shares:

A: 88% = 0.88

B: 12 % = 0.12

$v_0 = \begin{pmatrix}0.88\\ 0.12\end{pmatrix}$ initial market share vector

Transition matrix: constant transition rates in each time period

$$A = \begin{pmatrix}0.30 & 0.35\\ 0.70 & 0.65\end{pmatrix}$$

$$\underline{v_1} = A \cdot \underline{u_0} = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix} \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$$

$$= \begin{pmatrix} 0.30 \cdot 0.88 + 0.35 \cdot 0.12 \\ 0.70 \cdot 0.88 + 0.65 \cdot 0.12 \end{pmatrix} = \begin{pmatrix} 0.306 \\ 0.694 \end{pmatrix}$$

market share after one time period

$$\underline{v_n} = A^n \cdot \underline{u_0}$$

market share after n time periods

Compute A^n:

$$A = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix}$$

Eigenvalues:

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

$$\lambda^2 - 0.95\lambda - 0.05 = 0$$

$$\underline{\lambda = 1} \text{ or } \lambda = \underline{-0.05}$$

ok

(need $n=2$ eigenvalues)

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -0.05 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 1^n & 0 \\ 0 & (-0.05)^n \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Eigenvectors:

$\underline{\lambda = 1}$:
$$\begin{pmatrix} -0.70 & 0.35 \\ 0.70 & -0.35 \end{pmatrix} \to \begin{pmatrix} -0.7 & 0.35 \\ 0 & 0 \end{pmatrix}$$

$$-0.7x + 0.35y = 0$$

$$y = \frac{0.70x}{0.35} = 2x, \quad x \text{ free}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x \cdot \underline{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \qquad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

basis of E_1

ok

$\underline{\lambda = -0.05}$:
$$\begin{pmatrix} 0.35 & 0.35 \\ 0.70 & 0.70 \end{pmatrix} \to \begin{pmatrix} 0.35 & 0.35 \\ 0 & 0 \end{pmatrix}$$

$$0.35x + 0.35y = 0$$

$$x = -y, \quad y \text{ free}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

basis of $E_{-0.05}$

ok $\quad P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$

$\Downarrow$

$$A^m = P \cdot D^m \cdot P^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & (-0.05)^m \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\xrightarrow{m \to \infty} \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}$$

Markov chain: $\quad \mathbf{v_m} = A^m \cdot \mathbf{v_0} \to \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} \cdot \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$

$$= \begin{pmatrix} 1/3 \cdot 0.88 + 1/3 \cdot 0.12 \\ 2/3 \cdot 0.88 + 2/3 \cdot 0.12 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

Equilibrium state of the Markov chain