Plan

1 Vectors and vector operations
2 Span and linear independence
3 Vector spaces and dimension

## (1) Vectors and vector operations

<u>Defn:</u> An <u>n-vector</u> is a one-dimensional data structure

$$\underline{v} = (v_1, v_2, v_3, \ldots, v_n)$$

$$= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}$$

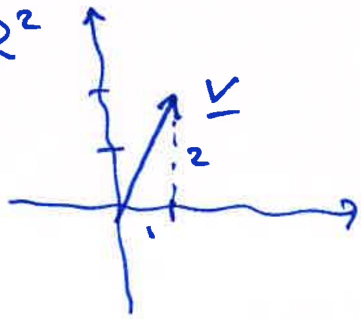
column vector

row vector

$\mathbb{R}^n$ = the space of all n-vectors
= n-dimensional space

<u>n=2:</u>

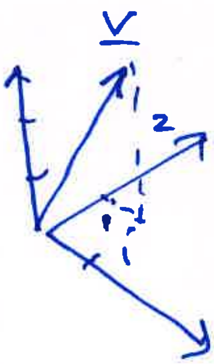
$\mathbb{R}^2$




$\underline{v} = (1,2)$

<u>n=3:</u>

$\mathbb{R}^3$




$\underline{v} = (1,1,2)$

Vector operations:

Addition: $\underline{v} + \underline{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)$
$= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

Scalar multiplication: $r \cdot \underline{v} = r \cdot (v_1, v_2, \ldots, v_n)$
$= (rv_1, rv_2, \ldots, rv_n)$

(2) Linear combination of vectors:

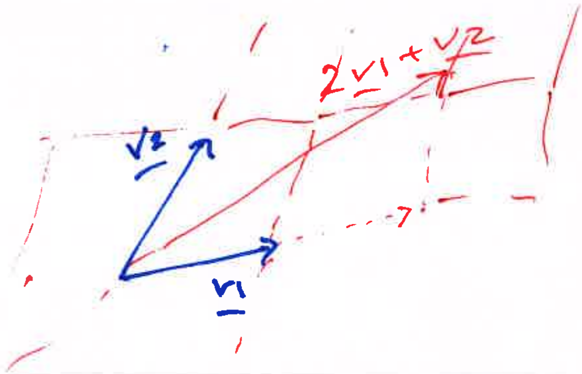
$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r} \rightsquigarrow c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + \cdots + c_r \cdot \underline{v_r}$
(where $c_1, c_2, \ldots, c_r$ are numbers)

Defn: The span of the vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}\}$ is

$$\text{span}(\underline{v_1}, \ldots, \underline{v_r}) = \{c_1 \underline{v_1} + c_2 \underline{v_2} + \cdots + c_r \underline{v_r} : c_1, c_2, \ldots, c_r \text{ are numbers}\}$$

Ex: $\underline{v_1} = (1,2,1)$
$\underline{v_2} = (3,-1,0)$

$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2}$
$= c_1 \cdot (1,2,1) + c_2 \cdot (3,-1,0)$
$= \underline{(c_1 + 3c_2, 2c_1 - c_2, c_1)}$

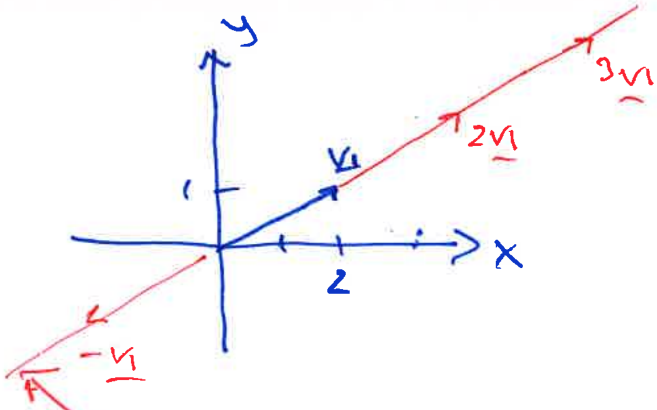



$\text{span}(\underline{v_1}, \underline{v_2})$ = the plane spanned by $\underline{v_1}, \underline{v_2}$ in $\mathbb{R}^3$

Ex:

i) $\underline{v_1} = (2,1)$:

$\text{span}(\underline{v_1}) = \{c_1 \cdot \underline{v_1}\}$




$\text{span}(\underline{v_1})$ is a line

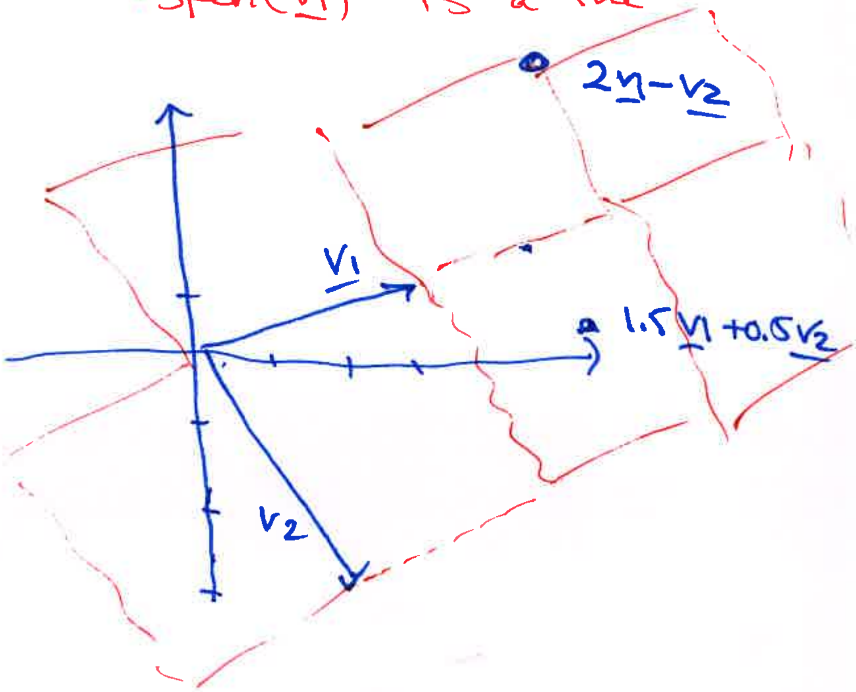
ii) $\underline{v_1} = (3,1)$ in $\mathbb{R}^2$

$\underline{v_2} = (2,-3)$

$\text{span}(\underline{v_1}, \underline{v_2})$

$= \{c_1 \underline{v_1} + c_2 \underline{v_2}\}$

$= \mathbb{R}^2$

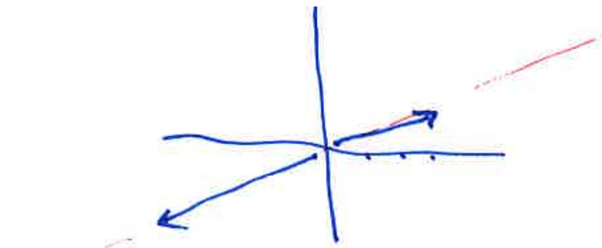



iii) $\underline{v_1} = (3,1)$

$\underline{v_2} = (-6,-2)$

$\underline{v_2} = -2 \cdot \underline{v_1}$

$\text{span}(\underline{v_1}, \underline{v_2})$ = a line in $\mathbb{R}^2$



## Linear independence:

$L = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ vectors in $\mathbb{R}^n$

**Defn:** The set $L$ of vectors is linearly dependent if at least one of the vectors is a linear combination of the others. Otherwise $L$ is a linearly independent set of vectors.

**Ex:** $\underline{v}_1 = (2,1,4)$
$\underline{v}_2 = (3,5,-1)$
$\underline{v}_3 = (5,-1,17)$

linearly dependent

$$\underline{v}_1 = x_2 \cdot \underline{v}_2 + x_3 \cdot \underline{v}_3 .$$

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -1 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3x_2 + 5x_3 \\ 5x_2 - x_3 \\ -x_2 + 17x_3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 5 & 2 \\ 5 & -1 & 1 \\ -1 & 17 & 4 \end{array}\right) \xrightarrow{2} \left(\begin{array}{cc|c} 1 & 39 & 10 \\ 5 & -1 & 1 \\ -1 & 17 & 4 \end{array}\right) \begin{array}{c} -5 \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 39 & 10 \\ 0 & -196 & -49 \\ 0 & 56 & 14 \end{array}\right) \begin{array}{c} \\ :49 \\ :14 \end{array} \rightarrow \left(\begin{array}{cc|c} 1 & 39 & 10 \\ 0 & -4 & -1 \\ 0 & 4 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 39 & 10 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$x + 39y = 10 \qquad x = 10 - \frac{39}{4} = 1/4$

$-4y = -1 \Rightarrow y = 1/4$

$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_2, \underline{v}_3)$

$c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3$

$= (\frac{1}{4} v_2 + \frac{1}{4} v_3) \cdot c_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3$

$\underline{v}_1 - \frac{1}{4}\underline{v}_2 - \frac{1}{4}\underline{v}_3 = \underline{0}$

$4\underline{v}_1 - \underline{v}_2 - \underline{v}_3 = \underline{0}$

$$\underline{v}_1 = \frac{1}{4}\underline{v}_2 + \frac{1}{4}\underline{v}_3$$

$\Updownarrow$

**Proposition:**

The vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}\}$ are <u>linearly independent</u> if and only if the vector equation

$$x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + \ldots + x_r \underline{v_r} = \underline{0}$$

only has the <u>trivial solution</u>.

<u>Explanation:</u>

$$x_1 \cdot \begin{pmatrix} v_{11} \\ \vdots \\ v_{1n} \end{pmatrix} + x_2 \begin{pmatrix} v_{21} \\ \vdots \\ v_{2n} \end{pmatrix} + \ldots + x_r \cdot \begin{pmatrix} v_{r1} \\ \vdots \\ v_{rn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

homogeneous linear system

(a)

<u>one solution</u>

$x_1 = x_2 = \ldots = x_r = 0$

a) linearly independent

(b)

<u>infinitely many solution</u>

(free variables)

b) linearly dependent

$\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}\}$ are :

## 3 Vector spaces and dimension

**Vector space:**

Ex: i) $V = \mathbb{R}^n$

ii) $V = \text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r) \subseteq \mathbb{R}^n$ subset

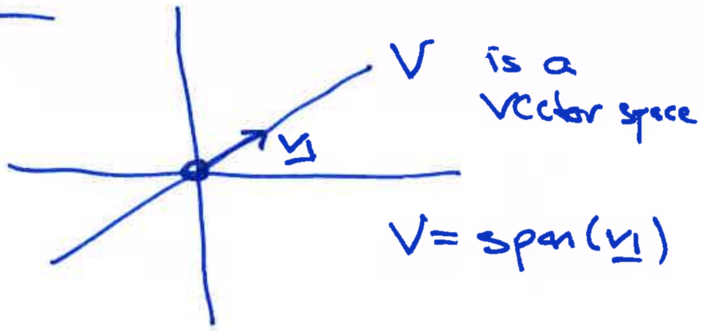
**Def:** A subset $V$ of $\mathbb{R}^n$ is a vector space if

i) $\underline{0}$ is in $V$

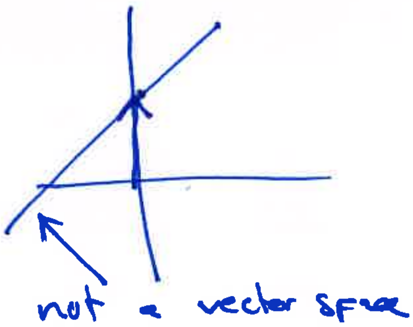
ii) Any linear combination of vectors in $V$ are in $V$

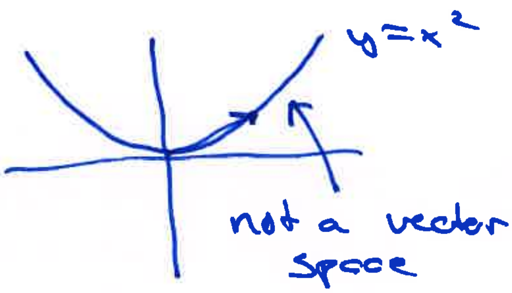
Note: $V = \text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r)$ always satisfy i) and ii).

Ex:




Ex:



a) $V = \text{Col}(A)$ where $A$ is a matrix

column space

$= \text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r)$

$$A = \left( \underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \dots \,\middle|\, \underline{v}_r \right)$$

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 5 \\ -1 \\ 17 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 6 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 6 \\ 0 & 7 & -7 \\ 0 & 7 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 6 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3$

found before: $\underline{v}_1 = \frac{1}{4}\underline{v}_2 + \frac{1}{4}\underline{v}_3$

$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_2, \underline{v}_3)$

$\underline{v}_3 = 4\underline{v}_1 - \underline{v}_2$

Conclusion:

$\{\underline{v}_1, \underline{v}_2\}$ is a base of $V$

$V_3$ is a linear combination of $\underline{v}_1, \underline{v}_2$

$\dim \text{Col}(A) = 2$

Defn: A base of a vector space $V$ is a set of vectors $B = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r\}$ such that

i) $\text{span}(B) = \text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r) = V$

ii) $B$ is a linearly independent set of vectors

The dimension of a vector space $V$ is the numbers of vectors in a base of $V$:

$\dim(V) = r$

Result: $V = \text{Col}(A)$ has $\dim V = \text{rk}(A)$

and vectors corresponding to pivot positions in $A$ form a base of $V$

Note: vectors not corresponding to pivots are linear combinations of the vectors in the base

Look at $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_r \underline{v_r} = \underline{0}$ : Homogeneous system with coeff. matrix $A$

$$\left( \underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_r} \right) = A$$

Example from previous page

$$\begin{pmatrix} 1 & -2 & 6 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 2x_2 + 6x_3 = 0$

$7x_2 - 7x_3 = 0$

$x_3 = t$ is free

$x_2 = x_3 = t$

$x_1 = 2x_2 - 6x_3$

$= 2x_3 - 6x_3 = -4x_3$

$= -4t$

$(x_1, x_2, x_3) = (-4t, t, t)$

$t=1$: $(-4, 1, 1) \to -4\underline{v_1} + \underline{v_2} + \underline{v_3} = 0$

$v_3 = 4\underline{v_1} - \underline{v_2}$

If we put one free variable $t=1$ (and all other free variables, if any, $=0$), we get how to write the corresponding column vector as a lin. comb. of the base!

## Plan

1. Parametrization of lines ~~and planes~~
2. Inner product, length and ~~projections~~
3. More examples

No time to cover:
- parametrization of planes
- projections

Continuation from Part 1: Some important vector spaces

i) Col(A): The column space of a matrix A
ii) Null(A): The null space of a matrix A
iii) Row(A): The row space of a matrix A

i) Col(A): $A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_r)$ $\quad Col(A) = span(\underline{v}_1, \dots, \underline{v}_r)$

Results: $\dim Col(A) = rk(A)$

Base of Col(A): column vectors corresponding to pivots

ii) Null(A): All solutions $\underline{x}$ of $A \cdot \underline{x} = \underline{0}$, i.e. the homogeneous linear system with coeff. matrix A

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix}$

$$2x + 3y + 5z = 0$$
$$x + 5y - z = 0$$
$$4x - y + 17z = 0$$

Solutions $(x, y, z) \rightsquigarrow$ Null(A)

$$\downarrow \quad \vdots \quad \downarrow$$

$$\begin{pmatrix} 1 & -2 & 6 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x - 2y + 6z = 0$$
$$7y - 7z = 0$$

z free

$z = t$

$$\frac{7y}{7} = \frac{7z}{7} \Rightarrow y = z = t$$

$$x = 2y - 6z = 2t - 6t = -4t$$

$(x,y,z) = (-4t, t, t) = t(-4,1,1) \Rightarrow$ all scalar multiples of $\underline{w}_1 = (-4,1,1)$

($t$ any number)

$\text{Null}(A) = \text{span}(\underline{w}_1)$

Base: $\{\underline{w}_1\}$

Dimension: $\dim \text{Null}(A) = 1$

**Result:** $\dim \text{Null}(A) = \#\text{ free variables} = n - \text{rk}(A)$

$A$ $m\times n$ matrix

If $t_1, t_2, \ldots, t_r$ are the free variables, then $\underline{x} = t_1 \cdot \underline{w}_1 + t_2 \cdot \underline{w}_2 + \ldots + t_r \underline{w}_r$

Base: $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_r\}$ (automatically linearly independent)

iii) Row(A):
$$A = \begin{pmatrix} \text{---}\; \underline{w}_1 \;\text{---} \\ \text{---}\; \underline{w}_2 \;\text{---} \\ \vdots \\ \text{---}\; \underline{w}_r \;\text{---} \end{pmatrix}$$

$\text{Row}(A) = \text{span}(\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_r)$

Ex: $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 5 & -1 \\ 4 & -1 & 17 \end{pmatrix}$

$\underline{w}_1 = (2,3,5)$

$\underline{w}_2 = (1,5,-1)$

$\underline{w}_3 = (4,-1,17)$

$\text{Row}(A) = \text{span}(\underline{w}_1, \underline{w}_2, \underline{w}_3)$

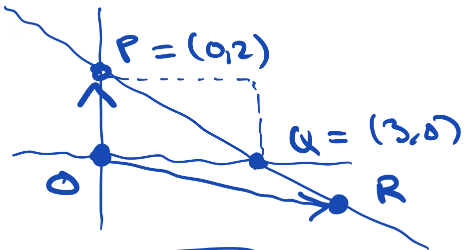
$$\downarrow \; \vdots \; \downarrow$$

$$\begin{pmatrix} 1 & -2 & 6 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$\{\underline{w}_1, \underline{w}_2\}$ is base of Row(A)

$$\boxed{\dim \text{Row}(A) = \text{rk}(A)}$$

## ① Parametrization of a line




$$\boxed{\begin{aligned} \vec{OR} &= \vec{OP} + \vec{PR} \\ &= \vec{OP} + t \cdot \vec{PQ} \end{aligned}}$$

Ex: $P = (0,2)$, $Q = (3,0)$

$\vec{OP} = (0,2)$

$\vec{PQ} = (+3, -2) = (3-0, 0-2)$

$(x,y) = (0,2) + t(3,-2)$

$= (3t, 2-2t)$

In general: The line through $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$:

$$\boxed{\begin{aligned}(x_1, x_2, \ldots, x_n) &= (p_1, p_2, \ldots, p_n) + t(q_1 - p_1, q_2 - p_2, \ldots) \\ &= (p_1 + t(q_1 - p_1), p_2 + t(q_2 - p_2), \ldots \\ &\qquad \ldots, p_n + t(q_n - p_n))\end{aligned}}$$

## ② Inner products, lengths (and projections)

Inner product: $\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + v_2 w_2 + \ldots + v_n \cdot w_n$

$\underline{v} = (v_1, v_2, \ldots, v_n)$ (the result is a number)

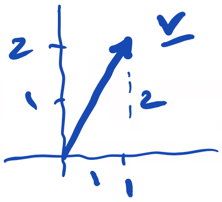
$\underline{w} = (w_1, w_2, \ldots, w_n)$ $\quad \underline{v} \cdot \underline{v} = v_1^2 + v_2^2 + \ldots + v_n^2$

(dot product)

i) $|\underline{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{\underline{v} \cdot \underline{v}}$

(the length of the vector $\underline{v}$)

Ex: $\underline{v} = (1, 2)$




$|\underline{v}| = \sqrt{1^2 + 2^2}$
$= \sqrt{5}$

Fact: The dot product is bilinear

Ex: $(\underline{v_1} + \underline{v_2}) \cdot (\underline{w_1} - 2\underline{w_2}) = \underline{v_1} \cdot \underline{w_1} + \underline{v_2} \cdot \underline{w_1}$
$+ \underline{v_1} \cdot (-2\underline{w_2}) + \underline{v_2} \cdot (-2\underline{w_2})$

$= \underline{v_1} \cdot \underline{w_1} + \underline{v_2} \cdot \underline{w_1} - 2\, \underline{v_1}\, \underline{w_2} - 2\, \underline{v_2} \cdot \underline{w_2}$

ii) <u>Distance between pts:</u>

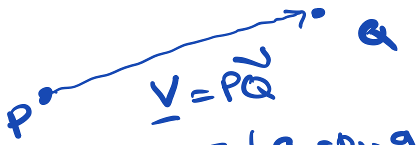
$P = (p_1, p_2, \ldots, p_n)$

$Q = (q_1, q_2, \ldots, q_n)$

$d(P,Q)$ = distance between $P$ and $Q$

$= |\vec{PQ}|$

$= \sqrt{(q_1-p_1)^2 + (q_2-p_2)^2 + \cdots + (q_n-p_n)^2}$




$\underline{v} = \vec{PQ} = (q_1-p_1, q_2-p_2, \ldots, q_n-p_n)$

③ <u>More examples:</u> Null(A)

<u>Ex:</u> $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \end{pmatrix} \begin{matrix} \\ ]-1 \\ ]-2 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{matrix} \\ \\ ]-1 \end{matrix}$

$\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$n = 4$ (# variables)

$\text{rk} A = 2$ (# pivots)

$\dim \text{Null}(A) = 4 - 2 = \underline{\underline{2}}$ (# free var's)

$\left.\begin{matrix} x_3 = s \\ x_4 = t \end{matrix}\right\}$ free var's

$\underline{x_1} + x_2 + x_3 + x_4 = 0$

$\underline{x_2} - 2x_3 = 0$

$x_2 = 2x_3 = \underline{2s}$

$x_1 + x_2 + x_3 + x_4 = x_1 + 2s + s + t = 0$

$\Rightarrow x_1 = \underline{-3s - t}$

<u>Null(A):</u>

$\underline{x} = (x_1, x_2, x_3, x_4) = (-3s - t, 2s, s, t)$

$= (-3s, 2s, s, 0) + (-t, 0, 0, t)$

$= \underline{s \cdot (-3, 2, 1, 0) + t \cdot (-1, 0, 0, 1)}$

<u>Basis:</u> $\{\underline{w_1}, \underline{w_2}\}$

$\underline{w_1} = (-3, 2, 1, 0)$

$\underline{w_2} = (-1, 0, 0, 1)$