GRA 6035/ELE 3781, lecture 13, 19 Nov 2021, Runar Ile

Plan
1. Intro. to difference equations
2. Linear first order diff.c. eq.s
3. Linear second order diff.c. eq.s
4. Systems of diff.c. eq.s
5. Stability of diff.c. eq.s.
6. Two exam problems (Jan. 2021, Nov. 2019)

1. Intro. to difference eq.s.

Ex You deposit 2000 into a bank account with 5% annual interest. Let y_t be the account balance after t years.

Then
$$\left.\begin{aligned} y_0 &= 2000 \\ y_1 &= 2000 \cdot 1.05 \\ y_2 &= 2000 \cdot 1.05^2 \\ &\vdots \\ y_t &= 2000 \cdot 1.05^t \\ y_{t+1} &= 2000 \cdot 1.05^{t+1} \end{aligned}\right\} \text{a sequence of numbers}$$

Note that
$$\underbrace{y_{t+1} - y_t}_{\Delta t} = 2000 \cdot 1.05^{t+1} - 2000 \cdot 1.05^t$$
$$= 2000 \cdot 1.05^t \cdot (1.05 - 1)$$
$$= 0.05 \cdot y_t$$

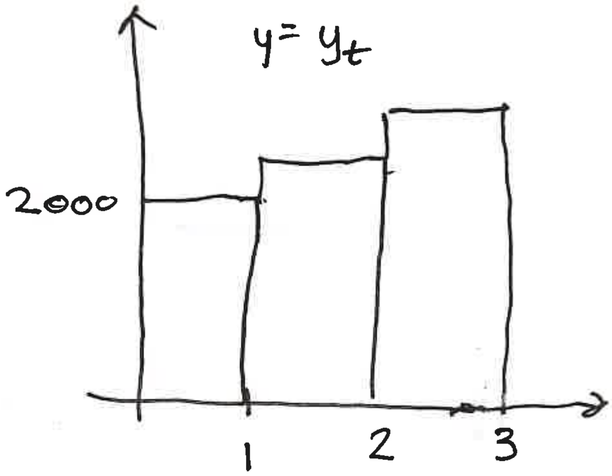
that is $y_{t+1} - 1.05\, y_t = 0$ a difference eq.

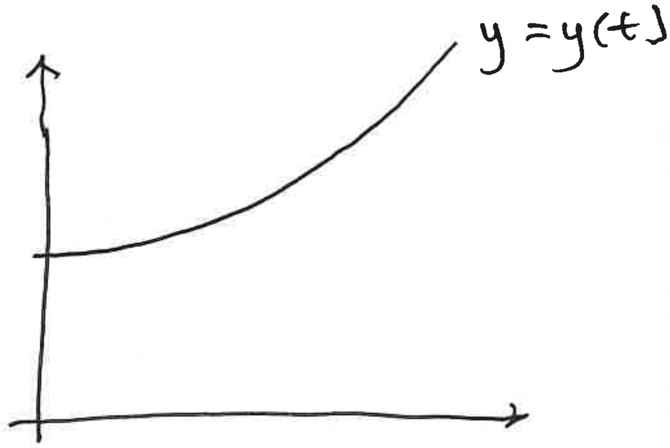
Has general solution: $y_t = C \cdot 1.05^t$

Analogous: Differential equation

$$y'(t) = 0.05 \cdot y(t)$$

General solution: $y(t) = C \cdot e^{0.05t}$





Note $e^{0.05} \approx 1.0513$

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2. Linear first order diffence eq.s.

Standard form $\quad y_{t+1} + a \cdot y_t = b_t \quad$ a is const. $\{b_t\}$ a seq.

If $b_t = 0$ for all t, then the diff.c. eq. is <u>homogeneous</u>.

Then the general solution is

$$y_t = y_t^h + y_t^p \qquad \text{"superposition"}$$

where y_t^h is the general sol. of the homog. eq. - involves 1 undetermined coeff.

y_t^p is a particular solutions.

Ex $\quad y_{t+1} - 1.05\, y_t = t + 10 \qquad (*)$

Then $\boxed{y_t^h = C \cdot 1.05^t}$

Guess $y_t^p = At + B$. Use $(*)$ to determ. A and B.

Then $y_{t+1}^p = A(t+1) + B$

so $A(t+1) + B - 1.05(At + B) = t + 10$

so $\boxed{-0.05A}\,t + (A - 0.05B) = \boxed{1}\,t + 10$

$$\begin{cases} -0.05A = 1 \\ A - 0.05B = 10 \end{cases} \quad \text{i.e.} \quad \begin{cases} A = -20 \\ B = -600 \end{cases}$$

So $\boxed{y_t^p = -20t - 600}$, and the gen. sol.

is $\underline{\underline{y_t = C \cdot 1.05^t - 20t - 600}}$

3. Linear second order difference eq.s.

<u>Standard form</u> $\quad y_{t+2} + a\, y_{t+1} + b y_t = f_t$

a, b are constants, $\{f_t\}$ sequence.

Again the gen. solution is $y_t = y_t^h + y_t^p$

The homog. gen. sol. has <u>two</u> undetermined constants.

EX $y_{t+2} - 7y_{t+1} + 12y_t = t \qquad (*)$

y_t^h Char. eq: $r^2 - 7r + 12 = 0$ so $r = 3$ or $r = 4$

Then $\boxed{y_t^h = C_1 \cdot 3^t + C_2 \cdot 4^t}$

[if double root r, then $y_t^h = C_1 r^t + C_2 \cdot t \cdot r^t$]

y_t^p Guess $y_t^p = At + B$ (same type as $f_t = t$)

Then $y_{t+1}^p = A(t+1) + B = At + A + B$

and $y_{t+2}^p = A(t+2) + B = At + 2A + B$

Use $(*)$: $A(t+2) + B - 7\big(A(t+1) + B\big) + 12(At + B) = t$

Collect t-terms and constants

Get $\boxed{6A}\,t\ (-5A + 6B) = \boxed{1}\,t\ \ 0$

$$\begin{cases} 6A = 1 \\ -5A + 6B = 0 \end{cases} \quad \text{get} \quad \begin{cases} A = \frac{1}{6} \\ B = \frac{5}{36} \end{cases}$$

Gen. sol. $y_t = C_1 \cdot 3^t + C_2 \cdot 4^t + \frac{t}{6} + \frac{5}{36}$

Start 9.00

4. Systems of difference eq. s

<u>Ex</u> Two sequences : $y_{1,t}$ and $y_{2,t}$

and two eq. s.

$$\begin{cases} y_{1,t+1} = 0.85\, y_{1,t} + 0.14\, y_{2,t} \\ y_{2,t+1} = 0.15\, y_{1,t} + 0.86\, y_{2,t} \end{cases}$$

a coupled system

Matrix form $\quad \mathbf{y}_{t+1} = \begin{bmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{bmatrix} \mathbf{y}_t$

where $\mathbf{y}_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix}$. Now we use matrix methods!

Find eigenvalues and eigenvectors.

$$\det \begin{bmatrix} 0.85-\lambda & 0.14 \\ 0.15 & 0.86-\lambda \end{bmatrix} = \lambda^2 - 1.71\lambda + 0.71 \overset{\text{eq.}}{=} 0$$

Get $\lambda = 1$, $\lambda = 0.71$ as eigenvalues.

Eigenvectors $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$:

<u>$\lambda = 1$</u> $\quad \begin{bmatrix} -0.15 & 0.14 \\ 0.15 & -0.14 \end{bmatrix} \big) +1 . \sim \begin{bmatrix} -0.15 & 0.14 \\ 0 & 0 \end{bmatrix}$

Get $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = t \begin{bmatrix} 14/15 \\ 1 \end{bmatrix}$ t is a free parameter

If $t = 15$ we get $\begin{bmatrix} 14 \\ 15 \end{bmatrix}$ as an eigenvector with eigenvalue 1

$\lambda = 0.71$ $\begin{bmatrix} 0.14 & 0.14 \\ 0.15 & 0.15 \end{bmatrix}$ gives solutions

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, e.g. $t=1$ gives $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

as eigenvector with eigenvalue 0.71.

Then the general solution is

$$\boxed{y_t = C_1 \cdot \begin{bmatrix} 14 \\ 15 \end{bmatrix} \cdot 1^t + C_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot 0.71^t}$$

i.e. $\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 14C_1 - C_2 \cdot 0.71^t \\ 15C_1 + C_2 \cdot 0.71^t \end{bmatrix}$

Always like this if you have n distinct eigenvalues

5. Stability of difference equations

Ex $y_{t+1} - 0.95 y_t = -100$

Rewrite $\Delta_t = y_{t+1} - y_t = -0.05 y_t - 100$

Then the number y_e (a constant!) is called an equilibrium state if $y_t = y_e$ substituted into the right hand side gives 0 : $-0.05 y_e - 100 = 0$

Solve eq: $y_e = \dfrac{100}{-0.05} = -2000$

If $y_0 = y_e = -2000$, then

$$y_1 - y_0 = -0.05 \cdot \underbrace{y_e}_{=-2000} - 100 = 0$$

so $y_1 = -2000$. Then

$$y_2 - y_1 = -0.05 \cdot \underbrace{y_1}_{=-2000} - 100 = 0$$

so $y_2 = y_1 = -2000$, and so on.

$\{y_t\}$ with $y_0 = -2000$ is a constant sequence (all are $= -2000$)

-2000 is an equilibrium state.

What if $y_0 = -1999$ or $y_0 = -2001$?

<u>Question</u> Will $y_t \xrightarrow[t \to \infty]{} -2000$?

Well, the general solution of diff. c. eq. is

$$y_t = C \cdot 0.95^t - 2000$$

$$\lim_{t \to \infty} y_t = C \cdot 0 - 2000 = \underline{\underline{-2000}}$$

If $\lim_{t \to \infty} y_t = y_e$ then

y_e is called stable.

If this is independent of constants, then y_e is globally asymptotically stable. (as in the example).

Video 13 for GRA 6035 / ELE 3781, 19 Nov 2021, Runar Ile

Plan
1. A bit more about stability.
2. Exam 18 Jan 2021, q. 3a
3. Exam 27 Nov 2019, q. 5

1. Stability

- have similar concepts for systems of diff. c. eq:
  - A equilibrium vector y_e. stable?

In fact, if $-1 < \lambda_i < 1$ for all eigenvalues then y_e is globally asymptotically stable.

Ex $y_{t+1} = A y_t$ is regular Markov chain.

2. Exam 2021 (Jan.)

Question 3.

(a) **(6p)** Solve the difference equation $y_{t+2} - y_{t+1} - 2y_t = 4t$, and find y_{17} when $y_0 = y_1 = 1$. 43%

(b) **(6p)** Determine whether $t^2y' + 2ty = 1$ is (i) separable, (ii) linear, (iii) exact. Use this to solve the differential equation in at least two different ways. 45%

(c) **(6p)** Find a linear second order differential equation with $y = 3e^{-2t} - 5e^t + 12e^{-3t}$ as solution. 33%

(d) **(6p)** Find a 3×3 matrix A such that $\mathbf{y}' = A\mathbf{y}$ has a solution $\mathbf{y} = (y, y', y'')$ with y as in (c). 3%

a) $y_t = y_t^h + y_t^p$

Plan:
1. Determine y_t^h by using char. eq.
2. Guess y_t^p similar to $4t$ and find coeff.
3. Use init. cond. to get two eq. which determine ~~the~~ the unknown coeff. (C_1, C_2)
4. We use the expression for y_t to calc y_{17}.

(1) Char. eq. $r^2 - r - 2 = 0$ so $\underline{r = -1}$ and $\underline{r = 2}$

$$y_t^h = C_1 \cdot (-1)^t + C_2 \cdot 2^t$$

(2) Guess $y_t^p = At + B$ and use the eq. to determine A and B.

$$y_{t+1}^p = A(t+1) + B = At + A + B$$

$$y_{t+2}^p = A(t+2) + B = At + 2A + B$$

Insert into diff. e. eq:

$$At + 2A + B - (At + A + B) - 2(At + B) = 4t$$

Collect t-terms and constants

$$\boxed{-2A}\,t + \boxed{A - 2B} = \boxed{4}\,t + \boxed{0}$$

$$\begin{cases} -2A = 4 \\ A - 2B = 0 \end{cases} \quad \text{get} \quad \begin{cases} A = -2 \\ B = -1 \end{cases}$$

so $y_t^p = -2t - 1$ and

$$y_t = C_1 \cdot (-1)^t + C_2 \cdot 2^t - 2t - 1$$

(3) Determine C_1 and C_2 from $\begin{cases} y_0 = 1 \\ y_1 = 1 \end{cases}$

$$1 = y_0 = C_1 \cdot (-1)^0 + C_2 \cdot 2^0 - 2 \cdot 0 - 1 = C_1 + C_2 - 1$$

$$1 = y_1 = C_1 \cdot (-1)^1 + C_2 \cdot 2^1 - 2 \cdot 1 - 1 = -C_1 + 2C_2 - 3$$

$$\begin{cases} c_1 + c_2 - 1 = 1 \\ -c_1 + 2c_2 - 3 = 1 \end{cases} \quad \text{solve and get} \quad \begin{cases} c_1 = 0 \\ c_2 = 2 \end{cases}$$

$$\boxed{y_t = 2 \cdot 2^t - 2t - 1}$$

(4) $y_{17} = 2 \cdot 2^{17} - 2 \cdot 17 - 1 = \underline{\underline{2^{18} - 35}} = \underline{\underline{262109}}$

3. Exam Nov 2019

Question 5.

Extra credit (6p) Find the particular solution of the system of difference equations that satisfies the given initial condition:

$$\mathbf{y}_{t+1} = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix} \cdot \mathbf{y}_t, \quad \mathbf{y}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Meaning $\underline{y}_{t+1} = \begin{bmatrix} y_{1,t+1} \\ y_{2,t+1} \\ y_{3,t+1} \end{bmatrix} = \begin{bmatrix} 4y_{1,t} + 6y_{3,t} \\ -y_{1,t} + 3y_{2,t} \\ y_{1,t} + y_{2,t} + 2y_{3,t} \end{bmatrix}$

– a coupled homogeneous system.

General (homog.) solution

$$\underline{y}_t = C_1 \cdot \underline{v}_1 \cdot \lambda_1^t + C_2 \cdot \underline{v}_2 \cdot \lambda_2^t + C_3 \cdot \underline{v}_3 \cdot \lambda_3^t$$

$\underline{v}_i$ is the eigenvector with eigenvalue λ_i

(1) Find eigenvalues: solve eq. $\det(A - \lambda I) = 0$

so $\det \begin{bmatrix} 4-\lambda & 0 & 6 \\ -1 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$

Get (after calculation) $(\lambda-4)\cdot\lambda\cdot(5-\lambda)=0$

so $\underline{\lambda=0}$ or $\underline{\lambda=4}$ or $\underline{\lambda=5}$

(2) Find eigenvectors for each eigenvalue. by Gaussian elimination

$\underline{\lambda=0}$: $\begin{bmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 4 & 0 & 6 \end{bmatrix} \big) -4.$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 0 & -4 & -2 \end{bmatrix} \big) +1. \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix} \big) +1.$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ solve $\begin{cases} z_1 + z_2 + 2z_3 = 0 \\ 4z_2 + 2z_3 = 0 \\ z_3 = t \text{ (free)} \end{cases}$

Get $\underline{z} = t\cdot\begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix}$, put $t=2$ and get

$\underline{v}_1 = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$

$\underline{\lambda=4}$ gives (by same procedure)

$\underline{z} = t\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ so $t=1$ gives $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda=5}$ gives $\underline{v}_3 = \begin{bmatrix} -6 \\ -3 \\ 1 \end{bmatrix}$

so $\boxed{\mathbf{y}_t = c_1 \cdot \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \cdot 0^t + c_2 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot 4^t + c_3 \cdot \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} \cdot 5^t}$

(3) Determine c_1, c_2 and c_3 by initial value $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $t=0$ $\quad \mathbf{y}_0 = c_1 \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \cdot 1 + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot 1 + c_3 \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} \cdot 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Gives matrix eq.

$$\begin{bmatrix} -3 & -1 & 6 \\ -1 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Gaussian elimination on

$$\left[\begin{array}{ccc|c} -3 & -1 & 6 & 1 \\ -1 & 1 & -3 & 1 \\ 2 & 0 & 1 & 1 \end{array}\right] \quad \text{and get}$$

$$\left.\begin{aligned} c_1 &= \frac{1}{10} \\ c_2 &= \frac{7}{2} \\ c_3 &= \frac{4}{5} \end{aligned}\right\} \text{gives } \mathbf{y}_t = \frac{7}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot 4^t + \frac{4}{5} \cdot \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} \cdot 5^t$$

for $t \geq 1$.

and $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$