Plan

1. Second order differential equations
2. Linear second order differential equations

Review:

First order differential equations
- separable
- exact
- linear

$y' = F(t,y)$

1 Second order differential equations

A differential equation that involves y'', but no higher derivatives, is called a second order differential equation.

Ex:

$$y'' = 12t - 4 \quad | \int \ldots dt$$

$$y' = \int 12t - 4 \, dt$$

$$= 12 \cdot \frac{t^2}{2} - 4t + C$$

$$y' = 6t^2 - 4t + C \quad | \int \ldots dt$$

$$y = \int 6t^2 - 4t + C \, dt$$

$$= 6 \cdot \frac{t^3}{3} - 4\frac{t^2}{2} + Ct + D$$

$$y = \underline{2t^3 - 2t^2 + Ct + D}$$

general solution

depends on two undetermined coefficients

(2) Linear second order differential equations

Defn: A linear second order differential equation is a diff.equation that can be written

$$\boxed{y'' + a(t)\cdot y' + b(t)\cdot y = h(t)}$$

We say that it is homogeneous if $h(t)=0$, and it has constant coefficients if $a(t)=a$ and $b(t)=b$ are constants.

We are going to explain how to find the general solution in the case with constant coefficients.

(a) Solution method in the homogeneous case

Ex: $y'' - 3y' + 2y = 0$

Characteristic equation: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r=2$ or $r=1$

$$y = C_1 \cdot e^{2t} + C_2 \cdot e^{t}$$

(general solution)

$r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2} = \underbrace{\frac{-a}{2}}_{\alpha} \pm \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\beta} \cdot \sqrt{-1}$$

$y'' + ay' + by = 0$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

$\Delta = a^2 - 4b$

$\Delta > 0$: two distinct roots $r_1 \neq r_2$

$y = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$

$\Delta = 0$: $r_1 = r_2 = -a/2$

$y = C_1 e^{-a/2 \cdot t} + C_2 t e^{-a/2 \cdot t}$

$\Delta < 0$: no real roots

$y = e^{\alpha t} \cdot (C_1 \cdot \cos(\beta t) + C_2 \sin(\beta t))$

Ex: $y'' + 6y' + 9y = 0$

$$r^2 + 6r + 9 = 0$$

$(\Delta = 0)$ $$r = \frac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$$

$$r_1 = r_2 = -3$$

$$\underline{\underline{y = C_1 \cdot e^{-3t} + C_2 t e^{-3t}}} = (C_1 + C_2 t)\, e^{-3t}$$

Ex: $y'' - 4y' + 5y = 0$

$$r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4} \cdot \sqrt{-1}}{2}$$

$(\Delta < 0)$ $$= 2 \pm 1 \cdot \sqrt{-1}$$

$$\underline{\alpha = 2} \quad \underline{\beta = 1}$$

$$\underline{\underline{y = e^{2t}(C_1 \cdot \cos t + C_2 \cdot \sin t)}}$$

Explanation of the characteristic eqn:

Look for solutions of the form $y = e^{rt}$ in $\boxed{y'' + ay' + by = 0}$

$y = e^{rt} = 1 \cdot e^{rt}$

$y' = e^{rt} \cdot r = r \cdot e^{rt}$

$y'' = r(e^{rt} \cdot r) = r^2 \cdot e^{rt}$

$r^2 e^{rt} + a(re^{rt}) + b \cdot (1 \cdot e^{rt}) = 0$

$r^2 e^{rt} + are^{rt} + be^{rt} = 0$

$(r^2 + ar + b) \cdot e^{rt} = 0$

$\underline{r^2 + ar + b = 0}$

char. eqn.

$y' + ay = 0$

$u = e^{\int a\,dt} = e^{at+C}$

int. factor: e^{at}

$y'e^{at} + aye^{at} = 0$

$(y \cdot e^{at})' = 0$

$ye^{at} = C$

$y = C \cdot e^{-at}$

Concl:

e^{rt} is a solution of $y'' + ay' + by = 0$

$\Updownarrow$

$r^2 + ar + b = 0$

(b) Solution method in the inhomogeneous case

In general: $\boxed{y'' + ay' + by = h(t)}$

Ex: $y'' - 3y' + 2y = e^{-t}$ inhomogeneous

Solution method: Superposition principle

General solution: $y = y_h + y_p$

y_h: the general solution of the homogeneous equation $y'' + ay' + by = 0$

y_p: a particular solution of $y'' + ay' + by = h(t)$

Ex: $y'' - 3y' + 2y = e^{-t}$

Superposition: $y = y_h + y_p = \underline{C_1 e^{2t} + C_2 e^{t} + \frac{1}{6} e^{-t}}$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 2, r = 1 \Rightarrow y_h = C_1 \cdot e^{2t} + C_2 e^{t}$

y_p: $y'' - 3y' + 2y = e^{-t}$

"Guess" $y = A \cdot e^{-t}$

$y' = -A e^{-t}$

$y'' = A e^{-t}$

$(Ae^{-t}) - 3(-Ae^{-t}) + 2(Ae^{-t}) = e^{-t}$

$\underbrace{(A + 3A + 2A)}_{=1} e^{-t} = e^{-t}$

$6A = 1 \Rightarrow A = \frac{1}{6} \Rightarrow y_p = \frac{1}{6} e^{-t}$

Method of undetermined coeff.:

How to "guess":

- should depend on parameters (A)
- should have the same form as $h(t), h'(t), h''(t)$
  $e^{-t} \quad -e^{-t} \quad e^{-t}$

If it doesn't work, try to multiply with t

Explanation: Superposition principle

$y'' + a(t)y' + b(t)y = h(t)$ — linear second order differential equation

Differential operator

$D = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t) \cdot \quad : \quad D(y(t)) = y'' + a(t)y' + b(t)y$

Ex: $y'' - 3y' + 2y = e^{-t}$

$D(y) = e^{-t}$ where $D = \frac{d^2}{dt^2} - 3 \cdot \frac{d}{dt} + 2 \cdot$

$D(t) = 0 - 3 \cdot 1 + 2t = -3 + 2t$

$D(e^{-t}) = (e^{-t})'' - 3 \cdot (e^{-t})' + 2 \cdot (e^{-t})$

$= e^{-t} - 3(-e^{-t}) + 2e^{-t} = 6e^{-t}$

D is linear: differential operator

$$\boxed{\begin{array}{l} (1)\ D(y_1 + y_2) = D(y_1) + D(y_2) \\ (2)\ D(c \cdot y) = c \cdot D(y) \end{array}}$$

Applications:

i) $D(e^{-t}) = 6e^{-t} \Rightarrow D(\frac{1}{6}e^{-t}) = \frac{1}{6}D(e^{-t})$

$= \frac{1}{6} \cdot 6e^{-t} = e^{-t}$

ii) $y'' - 3y' + 2y = 0 \qquad D = \frac{d^2}{dt^2} - 3 \cdot \frac{d}{dt} + 2 \cdot$

$r^2 - 3r + 2 = 0$

$r = 2, r = 1 \Rightarrow e^{2t}, e^{t}$ are solutions

$y = C_1 e^{2t} + C_2 e^{t}$ is the general solution

Linear property: $D(e^{2t}) = 0$, $D(e^{t}) = 0$ $\Rightarrow D(C_1 e^{2t} + C_2 e^{t})$

$= D(C_1 e^{2t}) + D(C_2 e^{t}) = C_1 \underbrace{D(e^{2t})}_{=0} + C_2 \cdot \underbrace{D(e^{t})}_{=0} = 0$ ✓

iii) Superposition principle:

$$y'' + a(t)y' + b(t)y = h(t) \quad = y_h + y_p$$

general solution

$$D(y) = h(t)$$

$$D = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t).$$

linear diff. operator

(a) Assume $y = y_h + y_p$, where

y_h: general solution of the homogeneous equation $D(y_h) = 0$

y_p: particular solution $D(y_p) = h(t)$

Then:

$$D(y) = D(y_h + y_p) = D(y_h) + D(y_p) = 0 + h(t) = h(t)$$

$\Rightarrow$ y is a solution

(b) Assume y is a solution, i.e. $D(y) = h(t)$

Then

$$D(y - y_p) = D(y) - D(y_p) = h(t) - h(t) = 0$$

y_p: a particular solution, i.e. $D(y_p) = h(t)$

$\Rightarrow$ $y - y_p$ is a solution of the homogeneous eqn.

$\Rightarrow$ $y - y_p = y_h$ $\Rightarrow$ $y = y_h + y_p$.

Ex: $y' - y = e^{2t}$ first order linear diff. eqn.

$$D(y) = e^{2t} \qquad D = \frac{d}{dt} - 1.$$

D has the linear property:

$$y = y_h + y_p = \underline{\underline{Ce^t + e^{2t}}}$$

Can use superposition

$$D(y_1 + y_2) = (y_1 + y_2)' - 1(y_1 + y_2) = y_1' + y_2' - y_1 - y_2 = (y_1' - y_1) + (y_2' - y_2) = D(y_1) + D(y_2)$$

y_h: $y' - y = 0$

$r - 1 = 0 \quad r = 1$

$y_h = C \cdot e^t$

$$D(c \cdot y_1) = (cy_1)' - 1 \cdot (cy_1) = cy_1' - c \cdot 1 \cdot y_1 = c\,(y_1' - y_1) = c \cdot D(y_1)$$

y_p: $y' - y = e^{2t}$

$y = Ae^{2t}$ $\quad A = 1$

$y' = 2Ae^{2t}$ $\quad y_p = 1 \cdot e^{2t}$

$2Ae^{2t} - Ae^{2t} = e^{2t}$

$D(e^{2t}) = 2e^{2t} - e^{2t} = e^{2t}$

$y_p = e^{2t}$

Summary: Linear differential eqn's

① $y' + a(t)y = b(t)$ first order linear

② $y'' + a(t)y' + b(t)y = h(t)$ second order linear

* Superposition always works : $y = y_h + y_p$
* Characteristic eqn: works if ① $a(t) = a$ constant
  ② $a(t) = a$, $b(t) = b$ constants

Plan

1. Equilibrium states and stability
2. Superposition principle

① Equilibrium states and stability:

Defn: A first order differential eqn. is called <u>autonomous</u> if it can be written $y' = F(y)$ ← no t in the diff. eqn.

Ex:
$$y' = 2y$$
$$y' = 5y \cdot (1 - y/10)$$

A <u>equilibrium state</u> of $y' = F(y)$ is value $y = y_e$ such that $F(y_e) = 0$

Ex: 1) $y' = 2y$ <u>Eq. states:</u> $2y = 0$

$y = 0 \Rightarrow \underline{\underline{y_e = 0}}$

2) $y' = 5y(1 - y/10)$

<u>Eq. states:</u> $\underline{5y}(1 - y/10) = 0$

$y = 0$ or $y = 10$

$5y = 0$ $\quad 1 - y/10 = 0$

$\Downarrow$

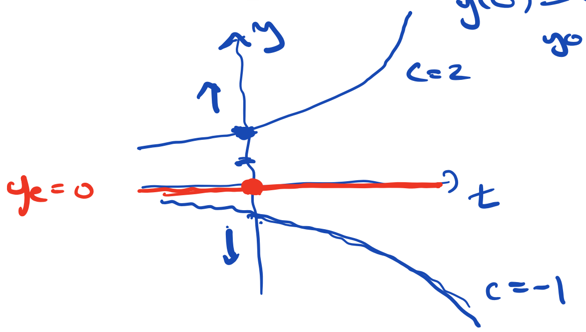
$y_e = \underline{0}$, $y_e = \underline{10}$

<u>$y' = 2y$:</u> $y = Ce^{2t}$

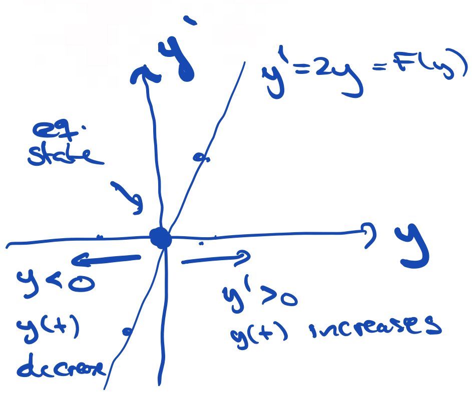
$y(0) = C \cdot e^0 = C$

$y_0 = C$

$y_e = 0$ is <u>unstable</u>



If $y_0 = 0$: $y = 0$



<u>phase diagram</u>

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Defn.: An eq. state $y = y_e$ is called

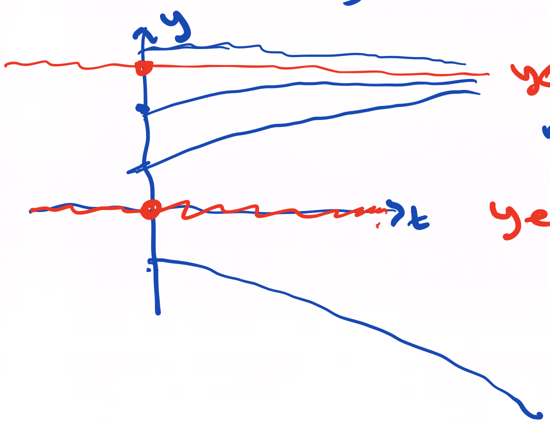
i) <u>stable</u> if y_0 close y_e means that $y(t)$ moves towards y_e

ii) <u>unstable</u> if y_0 — " — , — moves further away from y_e

iii) <u>globally asymptotically stable</u> if $y(t)$ moves towards y_e no matter which starting point y_0 you consider

Ex: $y' = 5y(1 - y/10)$ $\quad y_e = 0$, $y_e = 10$

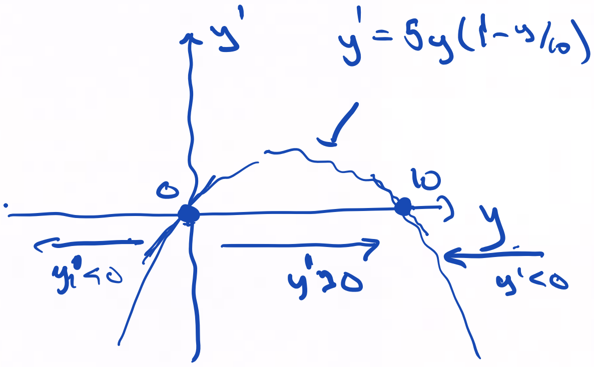
$= 5y - \frac{1}{2}y^2$



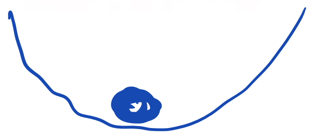
$y_e = 10$ <u>stable</u>

not globally asympt. stable

$y_e = 0$ <u>unstable</u>

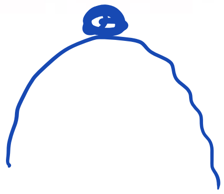


phase diagram



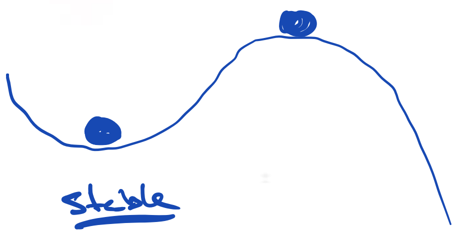
<u>stable</u>

globally asymptotically stable



<u>unstable</u>

<u>unstable</u>



<u>stable</u>

not globally asymptotically stable

<u>Stability Thm:</u>

If $y = y_e$ is an eq. state of $y' = F(y)$, then we have:

$F'(y_e) > 0 \Rightarrow y_e$ is <u>unstable</u>

$F'(y_e) < 0 \Rightarrow y_e$ is <u>stable</u>