Plan

1. First order differential equations
2. Separable differential equations
3. Exact differential equations

Mon Nov 01: Plenary Session 3
Problems from Lecture 7-9

Review: Quadratic functions

$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C \Rightarrow f'(\underline{x}) = 2A\underline{x} + B^T$

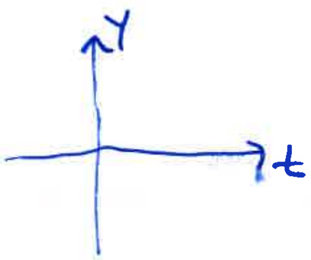
$H(f) = 2A$

A n×n symm. matrix, B 1×n matrix, C const.

① First order differential equations

Variable:

$y = y(t)$ function in one variable



t = time

$y(t)$: a quantity
$y'(t)$: rate of change

Differential equation of order one

An equation involving $y'(t)$ but not higher derivatives

Ex: $y' = 2y \Leftrightarrow y'(t) = 2 \cdot y(t)$

$y' + e^t y = 2\sqrt{t}$

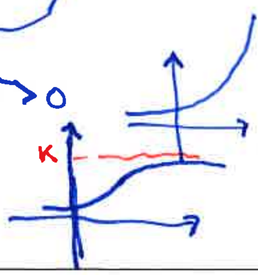
$\Leftrightarrow y' = 2\sqrt{t} - e^t y$

In general: $y' = F(t, y)$

Ex: $y' = r \cdot y \quad r > 0$

$y' = r \cdot y \cdot (1 - y/K)$

$r > 0, K > 0$



Ex: $y' = 2t$, $\underbrace{y(0) = 1}_{\text{initial condition}}$

$\Downarrow$

$$y = \int 2t\,dt = t^2 + C$$

General solution: $y(t) = t^2 + C$

$y(0) = 1$: $y(0) = 0^2 + C = 1$

$C = 1$

$\Downarrow$

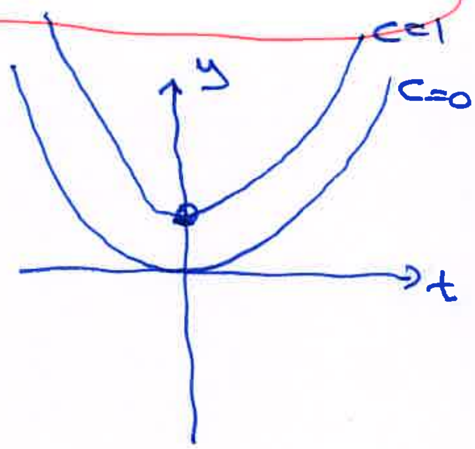
Particular solution: $y(t) = t^2 + 1$

A first order differential equation has general solution that involves one undetermined coefficient. ← general solution

A first order differential equation with one initial condition give a spesific value of the undetermined coefficient. ↖ particular solution

Ex: General solution: $y(t) = t^2 + C$

Each value of $C \Leftrightarrow$ one solution curve



2 Seperable differential equations

Defn: A first order differential equation is separable if it can be written

$$y' = f(t) \cdot g(y)$$

$y' = F(t,y)$ general first order diff. eqn

Ex:

1. $y' = 2y = \underbrace{2}_{f(t)} \cdot \underbrace{y}_{g(y)}$ ← separable

2. $y' = y+t$ ← not seperable

3. $y' = ry$ (r const.) "exponential growth" is seperable

4. $y' = ry(1 - y/K)$ (r, K const.) "logistic growth model" is seperable

Solution method:

$$y' = \underbrace{2}_{f(t)}\underbrace{y}_{g(y)} \quad | : y$$

$$\frac{1}{y} \cdot y' = 2 \quad | \int .. dt$$

$$\int \frac{1}{y} y' \, dt = \int 2 \, dt$$

$$\int \frac{1}{y} dy = \int 2 \, dt$$

$$\ln|y| + C_1 = 2t + C_2$$

$$\ln|y| = 2t + (C_2 - C_1)$$

$$\ln|y| = 2t + K$$

implicit solution

$$y' = f(t) \cdot g(y) \quad | : g(y)$$

$$\frac{1}{g(y)} \cdot y' = f(t) \quad | \int .. dt$$

$$\int \frac{1}{g(y)} y' \, dt = \int f(t) \, dt$$

$$\boxed{\int \frac{1}{g(y)} dy = \int f(t) dt}$$

implicit solution

$y' dt = dy$: substitution $y = y(t)$, $dy = y' \cdot dt$

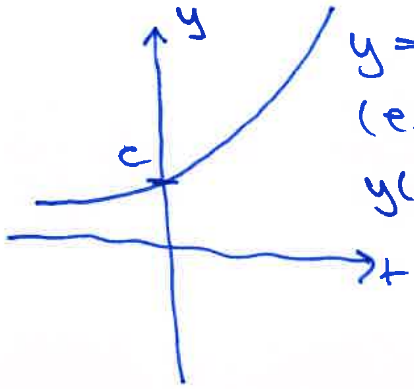
$$e^{\ln|y|} = e^{2t+K}$$

$$|y| = e^{2t}\cdot e^{K}$$

$$y = \pm e^{2t}\cdot e^{K}$$

$$\underline{\underline{y = C\cdot e^{2t}}}$$ general solution

(explicit form) $(C = \pm e^{K})$



$y = C\cdot e^{2t}$

(exponential fu.)

$y(0) = C$

<u>Ex:</u> $e^{2y}\cdot y' = te^{t}$ $\quad y' = f(t)\cdot g(y)$

$$y' = \underbrace{te^{t}}_{f(t)}\cdot\underbrace{e^{-2y}}_{g(y)}$$ <u>separable</u>

$$e^{2y}\cdot y' = te^{t}$$

$$\int e^{2y} y'\,dt = \int te^{t}dt$$

$$\boxed{\begin{matrix} u = e^{t} & v = t \\ u' = e^{t} & v' = 1 \end{matrix}}$$

$$\int e^{2y}\,dy = \int te^{t}dt \leftarrow \text{integration by parts}$$

$$\int u'v\,dt = uv - \int uv'\,dt$$

$$\tfrac{1}{2}e^{2y} = te^{t} - \int e^{t}\,dt$$

$$\tfrac{1}{2}e^{2y} = te^{t} - e^{t} + C \quad |\cdot 2$$

$$e^{2y} = 2te^{t} - 2e^{t} + 2C$$

$$\frac{2y}{2} = \ln(2te^{t} - 2e^{t} + 2C) = \underline{\ln(2\cdot(te^{t} - e^{t} + C))}$$

$$y = \frac{\ln[2\cdot(te^{t} - e^{t} + C)]}{2} = \frac{\ln 2 + \ln(te^{t} - e^{t} + C)}{2}$$ //

(3) Exact differential equations

Defn: A first order differential equation is exact if it can be written

$$p(t,y) + q(t,y)\cdot y' = 0$$

such that $p(t,y) = h'_t$ and $q(t,y) = h'_y$ for some function $h = h(t,y)$.

Solution: $h(t,y) = C$ — general solution in implicit form

Ex: $\underbrace{t^2}_{p} + \underbrace{y^2}_{q}\cdot y' = 0$

Want: $h'_t = t^2$, $h'_y = y^2$

(1) Look at $h'_t = t^2$:

$h = \int t^2\,dt = \frac{1}{3}t^3 + C(y)$

(2) Look at $h'_y = y^2$:

$h'_y = 0 + C'(y) = y^2$

$C'(y) = y^2$

$C(y) = \frac{1}{3}y^3 + C$

$h = \frac{1}{3}t^3 + \frac{1}{3}y^3 + C$

$\Downarrow$

yes, the diff. equ. is exact

and: the general solution is given by

$h = C$

$\frac{1}{3}t^3 + \frac{1}{3}y^3 + C_1 = C_2$

$\frac{1}{3}t^3 + \frac{1}{3}y^3 = C$ implicit solution

$\frac{1}{3}y^3 = C - \frac{1}{3}t^3 \Rightarrow y^3 = 3C - t^3 \Rightarrow y = \sqrt[3]{3C - t^3}$ (explicit form)

Remarks:

i) $t^2 + y^2 \cdot y' = 0$ exact $\Rightarrow$ $y = \sqrt[3]{K - t^3}$

Separable

$$y^2 \cdot y' = -t^2$$

$$y' = \underbrace{-t^2}_{f(t)} \cdot \underbrace{\frac{1}{y^2}}_{g(y)}$$

$$y^2 \cdot y' = -t^2$$

$$\int y^2 \, dy = \int -t^2 \, dt$$

$$\frac{1}{3} y^3 = -\frac{1}{3} t^3 + C \quad | \cdot 3$$

$$y^3 = -t^3 + 3C$$

$$y = \sqrt[3]{-t^3 + 3C} = \sqrt[3]{K - t^3}$$

ii) Exact: $h = \frac{1}{3}t^3 + \frac{1}{3}y^3 = C$ implicit derivation $\frac{d}{dt}(-)$

$$\boxed{\underbrace{t^2}_{h'_t} + \underbrace{y^2}_{h'_y} \cdot y' = 0}$$

iii) Test for exactness: $p(t,y) + q(t,y) \cdot y' = 0$ is exact

$\Updownarrow$

$$p'_y = q'_t$$

Ex: $t^2 + y^2 \cdot y' = 0$

$(t^2)'_y = 0 \quad (y^2)'_t = 0 \leftarrow$ Yes, it is exact.

Ex: $y' = \dfrac{3t^2 - y^2}{2ty} \quad | \cdot 2ty$ not separable

exact? $\quad p + q \cdot y' = 0$

$2ty \cdot y' = 3t^2 - y^2$

$\underbrace{(y^2 - 3t^2)}_{p} + \underbrace{2ty}_{q} \cdot y' = 0$

$h'_t + h'_y \cdot y' = 0$

$$\boxed{\begin{aligned} h'_t &= y^2 - 3t^2 \\ h'_y &= 2ty \end{aligned}}$$

① $h = \int y^2 - 3t^2 \, dt$

$= y^2 t - t^3 + C(y)$

② $h'_y = (y^2 t - t^3 + C(y))'_y$

$= 2yt - 0 + C'(y)$

$= 2yt + C'(y) = 2ty$

$C'(y) = 0$

$C(y) = C$

Cond: Yes, it is exact and

$h = y^2 t - t^3 + 0 = C$

$ty^2 - t^3 = C$

$\dfrac{ty^2}{t} = \dfrac{C + t^3}{t}$

$y^2 = \dfrac{C + t^3}{t}$

$y = \pm \sqrt{\dfrac{C + t^3}{t}}$

$y' = \dfrac{3t^2 - y^2}{2ty}$, $y(1) = 2$: General solution

$y = \pm \sqrt{\dfrac{C + t^3}{t}}$

$y(1) = 2$: $\quad 2 = \pm \sqrt{\dfrac{C + 1}{1}} = (+)\sqrt{C + 1}$

$4 = C + 1 \quad C = 3$

Particular solution

$y = \sqrt{\dfrac{3 + t^3}{t}}$

Plan

1 Linear first order differential equations
2 Superposition principle

① Linear first order differential equations

Defn: A first order differential equation is linear if it can be written

$$\boxed{y' + a(t)\cdot y = b(t)} \quad \left(\iff y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y} \right)$$

Ex: $y' = y + t \iff \boxed{y' - y = t}$ is linear, $a(t) = -1$, $b(t) = t$

$e^{2y}\cdot y' = te^t$ is not linear

$y' = te^t \cdot e^{-2y}$

Solution method: Integrating factor

Ex: $y' - y = t \quad | \cdot u \leftarrow$ integrating factor

choose $u = u(t)$ such that $u' = -u$

$uy' - uy = ut$

$uy' + u'y = ut$

$(uy)' = ut$

$\int (uy)'\,dt = \int u \cdot t\,dt$

$uy = \int t\cdot u(t)\,dt$

$y = \frac{1}{u(t)} \int t\cdot u(t)\,dt$ ✓

Formula: $u(t) = e^{\int a(t)dt}$

$\Rightarrow u'(t) = e^{\int a(t)dt} \cdot a(t)$

$= u(t) \cdot (-1)$

$= -u(t)$

Ex: $a(t) = -1 \Rightarrow \int -1\,dt = -t + C \Rightarrow$ Int. factor $u = e^{-t+C}$

$u = e^{-t}$

$$y' - y = t \quad | \cdot e^{-t}$$

$$e^{-t}y' - e^{-t}y = te^{-t}$$

$$(e^{-t}y)' = te^{-t}$$

$$e^{-t} \cdot y = \int te^{-t}\,dt$$

$$y = e^{t} \int te^{-t}\,dt$$

Integration by parts: $\int u'v\,dt = uv - \int uv'\,dt$

$u = -e^{-t}$ $v = t$

$u' = e^{-t}$ $v' = 1$

$$= e^{t}\left(-e^{-t} \cdot t - \int -e^{-t} \cdot 1\,dt\right)$$

$$= e^{t}\left(-te^{-t} + (-e^{-t}) + C\right)$$

$$= e^{t}\left(-te^{-t} - e^{-t} + C\right)$$

$$y = -t - 1 + Ce^{t}$$

general solution (explicit form)

In general:

$$y' + a(t)y = b(t) \quad | \cdot u$$

$$uy' + u \cdot a(t)y = b(t) \cdot u(t)$$

$$(u \cdot y)' = b(t) \cdot u(t)$$

$$u \cdot y = \int b(t) \cdot u(t)\,dt$$

$$\boxed{y = \frac{1}{u(t)} \int b(t) \cdot u(t)\,dt}$$

Integrating factor:

$$\boxed{u = e^{\int a(t)\,dt}}$$