Solutions: Problem Set C

1. See python code (on web page)

2. $\max/\min \int_0^3 \ln(y' + y + e^{-t})\,dt$ when $\begin{cases} y(0) = 3 \\ y(3) = 5e^{-3} \end{cases}$

Alt A: Variational calculus

$F = \ln(y' + y + e^{-t}) = \ln(u), \quad u = y' + y + e^{-t}$

$F'_y = \frac{1}{u} \cdot u'_y = \frac{1}{u} \cdot 1 = \frac{1}{u}$

$F'_{y'} = \frac{1}{u} \cdot u'_{y'} = \frac{1}{u} \cdot 1 = \frac{1}{u}$

$\frac{d}{dt}(F'_{y'}) = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d}{dt}(u^{-1}) = -1 \cdot u^{-2} \cdot u'_t$

$= -\frac{1}{u^2}(y'' + y' - e^{-t})$

Euler eqn: $F'_y - \frac{d}{dt}(F'_{y'}) = 0$

$\frac{1}{u} + \frac{1}{u^2} \cdot (y'' + y' - e^{-t}) = 0 \quad / \cdot u^2$

$u + 1 \cdot (y'' + y' - e^{-t}) = 0$

$y' + y + e^{-t} + y'' + y' - e^{-t} = 0$

$$y'' + 2y' + y = 0$$

lin. second order homogeneous eqn.

Ch. eqn: $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r_1 = r_2 = -1$

$$y = y_h = c_1 e^{-t} + c_2 t e^{-t}$$

$$y(t) = \underline{c_1 e^{-t} + c_2 t e^{-t}}$$

Initial cond:

$y(0) = 2$: $c_1 \cdot e^0 + c_2 \cdot 0 \cdot e^0 = 2$

$c_1 = 2$

$y(3) = 5e^{-3}$: $c_1 \cdot e^{-3} + c_2 \cdot 3 \cdot e^{-3} = 5e^{-3}$

$2e^{-3} + 3c_2 e^{-3} = 5e^{-3}$

$2 + 3c_2 = 5$

$3c_2 = 3$

$c_2 = 1$

Solution of Euler + initial cond:

$$y = 2e^{-t} + 1 \cdot t e^{-t} = \underline{(t+2)e^{-t}}$$

$F''_{yy} = -\frac{1}{u^2}$ $F''_{yy'} = -\frac{1}{u^2}$

$F''_{y'y} = -\frac{1}{u^2}$ $F''_{y'y'} = -\frac{1}{u^2}$

$H(F)$ neg. semidefin

Since $(y, y') \rightsquigarrow F(y, y')$ is concave, so the solution $y^* = \underline{(t+2)e^{-t}}$ is a $\underline{\text{max}}$

Max value:

$$y = (t+2)e^{-t}$$

$$y' = 1 \cdot e^{-t} + (t+2)e^{-t} \cdot (-1)$$

$$= -(t+1)e^{-t}$$

II

$$\int_0^3 \ln(y' + y + e^{-t})\, dt$$

$$= \int_0^3 \ln\left((t+2)e^{-t} - (t+1)e^{-t} + e^{-t}\right) dt$$

$$= \int_0^3 \ln(2e^{-t})\, dt = \int_0^3 \ln(2) + \ln(e^{-t})\, dt$$

$$= 3\ln(2) + \int_0^3 -t\, dt$$

$$= 3\ln(2) - \left[\tfrac{1}{2}t^2\right]_0^3$$

$$= 3\ln(2) - 9/2$$

Alt B: We can also reformulate to an optimal control pb:

$$u = y'$$

$$\downarrow$$

$$\max/\min \int_0^3 \ln(u + y + e^{-t})\,dt \quad \text{wh} \begin{cases} y(0) = 2 \\ y(3) = 5e^{-3} \\ y' = u \end{cases}$$

To find candidate pts (which gives max by the same argument as in Alt A), we use Hamiltonian + max min principle:

$$H = p_0 \cdot \ln(u + y + e^{-t}) + p \cdot u$$

i) $p_0 = 0$ or $p_0 = 1$, with $(p_0, p) \neq (0,0)$

ii) $H'_u = 0$

iii) $p'(t) = -H'_y$

<u>case $p_0 = 1$:</u> $H = \ln(u + y + e^{-t}) + p \cdot u$

$$H'_u = \frac{1}{u + y + e^{-t}} \cdot 1 + p = 0$$

$$H'_y = \frac{1}{u + y + e^{-t}} \cdot 1$$

$$\Rightarrow p' = -H'_y = -\frac{1}{u + y + e^{-t}} = p$$

$\boxed{\begin{array}{l} \underline{p' = p}: \\ p(t) = C \cdot e^{t} \end{array}}$ ← first order lin. diff. eqn. in p, $r - 1 = 0$ gives $r = 1$ and $p = C \cdot e^{t}$

$$H'_u = \frac{1}{u+y+e^{-t}} + p = 0$$

$$\frac{1}{u+y+e^{-t}} + Ce^{t} = 0 \quad \Big| \cdot (u+y+e^{-t})$$

$$1 + Ce^{t}(u+y+e^{-t}) = 0$$

$$u+y+e^{-t} = \frac{-1}{Ce^{t}}$$

$\underline{u=y'}$:

$$y'+y = -\frac{1}{C}e^{-t} - e^{-t} = D \cdot e^{-t} \qquad \boxed{D = -1-\frac{1}{C}}$$

$$y = y_h + y_p:$$

$\underline{y_h}$:

$$y'+y=0$$

$$r+1=0 \qquad r=-1 \Rightarrow y_h = Be^{-t}$$

$\underline{y_p}$:

$$y'+y = De^{-t}$$

$$y'+y=0 \quad \text{does not work}$$

$$y = Ae^{-t}$$
$$y' = -Ae^{-t}$$

$$y = Ate^{-t}$$
$$y' = Ae^{-t} + Ate^{-t}\cdot(-1) = (A-At)e^{-t}$$

$$(A-\cancel{At})e^{-t} + \cancel{Ate^{-t}} = De^{-t}$$

$$Ae^{-t} = De^{-t}$$

$$A = D$$

$$y_p = De^{-t}$$

<u>This gives</u>: $y = y_h + y_p = Be^{-t} + Dte^{-t}$

$$= \underline{(B+Dt)e^{-t}}$$

This is the same solution as in Alt A, so initial cond. would again give $y^* = \underline{(t+2)e^{-t}}$

<u>Case $p_0 = 0$:</u> $H = p \cdot u$

$H'_u = p = 0 \Rightarrow p(t) = 0$

(contradiction, since $(p_0, p) \neq (0,0)$).

$\Downarrow$

$y^* = \underline{(t+2)e^{-t}}$ is max