Solution: Key problems – Lecture 9

1. a) $$f'(x) = \frac{p\cdot a}{1+ax} + \frac{(-q)b}{1-bx} = \frac{pa(1-bx)-bq(1+ax)}{(1+ax)(1-bx)}$$

$$= \frac{ap-bq-ab(p+q)x}{(1+ax)(1-bx)} = \frac{ap-bq-abx}{(1+ax)(1-bx)} = 0$$

$$\Updownarrow$$

$$ap-bq = abx$$

$$x = \underline{\frac{ap-bq}{ab}} > 0 \qquad \text{since } ap-bq>0$$

$$f''(x) = \frac{-pa^2}{(1+ax)^2} + \frac{-qb^2}{(1-bx)^2} < 0 \implies f \text{ concave}$$

$$\underline{\underline{x^* = \frac{ap-bq}{ab} \quad \text{max point}}}$$

b) $\left.\begin{array}{l} a=2, b=1 \\ p=0.40 \; q=0.60 \end{array}\right\}$ $$x^* = \frac{2\cdot 0.4-1\cdot 0.6}{2\cdot 1} = \frac{0.2}{2} = \underline{\underline{0.10}} \quad \text{max point}$$

$$f^* = f(0.10) = 0.40\ln(1+2\cdot 0.10) + 0.60\cdot\ln(1-0.10)$$
$$= 0.40\ln(1.2) + 0.60\cdot\ln 0.90$$
$$\simeq \underline{\underline{0.0097}} \quad \text{max value}$$

c) $$\frac{df^*(p)}{dp} = \frac{\partial f}{\partial p}(x=x^*) \qquad \frac{\partial f}{\partial p} = \ln(1+ax)-\ln(1-bx) \text{ since } q=1-p$$

$$= \ln\left(1+a\cdot\frac{ap-bq}{ab}\right) - \ln\left(1-b\cdot\frac{ap-bq}{ab}\right)$$

$$= \ln\left(1+\frac{ap-bq}{b}\right) - \ln\left(1-\frac{ap-bq}{a}\right)$$

$\left.\begin{array}{l} a=2 \\ b=1 \\ p=0.40 \\ x^*=0.10 \end{array}\right\}$ $$= \ln(1+2\cdot 0.10) - \ln(1-1\cdot 0.10) = \ln(1.2)-\ln(0.9) \simeq 0.2877$$

Estimate for $p=0.43$:

$$f^*(0.43) \simeq f^*(0.40) + 0.03\cdot\frac{df}{dp} \simeq 0.0097 + 0.03\cdot 0.2877$$
$$= \underline{\underline{0.0183}}$$

2. max $f(x,y,z) = 4x^3 - 2y^3 + z^3$ when $x^3 + y^3 + z^3 \le 8$

a) $L = 4x^3 - 2y^3 + z^3 - \lambda(x^3 + y^3 + z^3)$

FOC:

$L'_x = 12x^2 - \lambda \cdot 3x^2 = 0 \qquad 3x^2(4-\lambda) = 0 \qquad x = 0 \text{ or } \lambda = 4$

$L'_y = -6y^2 - \lambda\, 3y^2 = 0 \qquad 3y^2(-2-\lambda) = 0 \qquad y = 0 \text{ or } \cancel{\lambda = -2}$

$L'_z = 3z^2 - \lambda \cdot 3z^2 = 0 \qquad 3z^2(1-\lambda) = 0 \qquad z = 0 \text{ or } \lambda = 1$

CSC: $\lambda \ge 0$

C: $x^3 + y^3 + z^3 = 8$ if $\lambda > 0$

Candidate pts:

i) $x = y = z = 0$, $x^3 + y^3 + z^3 = 0 < 8 \Rightarrow \lambda = 0 \Rightarrow \boxed{(0,0,0;0),\ f = 0}$

ii) $x = y = 0$, $z \ne 0$ and $\lambda = 1$:
$x^3 + y^3 + z^3 = z^3 = 8 \Rightarrow z = 2$ $\qquad \boxed{(0,0,2;1),\ f = 8}$

iii) $y = z = 0$, $x \ne 0$ and $\lambda = 4$:
$x^3 + y^3 + z^3 = 8 \Rightarrow x = 2$ $\qquad \boxed{(2,0,0;4),\ f = 32}$

iv) $\lambda = 4$ and $\lambda = 1$: impossible

Best candidate point: $(x,y,z;\lambda) = (2,0,0;4)$ $\quad f = 32$

b) There is no max since $f \to \infty$ when $x = 0$, $y \to -\infty$ and $z = 0$. For example,

$$f(0,-3,0) = 81 > 32$$

Moreover $(0,y,0)$ is admissible when $y \le 2$.

3. max $f(x,y,z) = 2x^2 - 4y^2 - 2z^2$ when $x^4+y^4+z^4 \le 16$

a) $x^4+y^4+z^4 \le 16$ defines a bounded set, since $-2 \le x,y,z \le 2$.

$\Downarrow$ EVT

there is a max.

b) NDCQ: a) $x^4+y^4+z^4 < 16$: no condition

b) $x^4+y^4+z^4 = 16$:

$\text{rk } J = \text{rk}\,(4x^3\ 4y^3\ 4z^3) = 1$

NDCQ fails if $x=y=z=0$, not adm.

NDCQ holds for all adm. pts.

c) From a), b) it follows that there is a max at a regular candidate pt, with FOC + C + CSC satisfied.

$L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4+y^4+z^4)$

FOC:

$L'_x = 4x - \lambda \cdot 4x^3 = 0$ $\quad 4x(1-\lambda x^2) = 0$

$L'_y = -8y - \lambda \cdot 4y^3 = 0$ $\quad 4y(-2-\lambda y^2) = 0$

$L'_z = -4z - \lambda \cdot 4z^3 = 0$ $\quad 4z(-1-\lambda z^2) = 0$

C + CSC:

a) $x^4+y^4+z^4 < 16$: $\lambda = 0 \Rightarrow x=y=z=0$ $\quad \boxed{(0,0,0;0) \quad f=0}$

b) $x^4+y^4+z^4 = 16$: $\lambda \ge 0$

$x=0$ or $\lambda = 1/x^2$

$y=0$ or $\lambda = -2/y^2 < 0 \Rightarrow y=0$

$z=0$ or $\lambda = -1/z^2 < 0 \Rightarrow z=0$

$x=0 \Rightarrow (x,y,z)=(0,0,0)$ not adm. impossible.

$\lambda = 1/x^2 \Rightarrow x^2 = 1/\lambda \Rightarrow x = \pm\sqrt{1/\lambda}$

$x^4+y^4+z^4 = (1/\lambda)^2 + 0 + 0 = 16 \Rightarrow \lambda^2 = 1/16 \Rightarrow \lambda = 1/4$

$\Rightarrow \boxed{(\pm 2, 0, 0; 1/4) \quad f=8}$ Best cand. pt $\Rightarrow$ $\underline{f_{max} = 8}$

$(x^*, y^*, z^*) = \underline{(\pm 2, 0, 0)}$

d) max $f(x,y,z) = ax^2 - 4y^2 - 2z^2$ when $g(x,y,z) = x^4 + y^4 + z^4 - b = 0$

Note: 1) $a = 2, b = 16$: $f^*(2,16) = 8$
$x^*(2,16) = \pm 2 \quad y^* = z^* = 0 \quad \lambda^* = 1/4$

2) the problem has a max which is regular cond. pt. as long as $a, b > 0$.

Envelope thm:

i) $a = 2, b = 20$: $f^*(2,20) = 8 + 4 \cdot \frac{df^*}{db} = 8 + 4 \cdot \underset{\lambda^*}{\frac{1}{4}} = \underline{\underline{9}}$

ii) $a = 1, b = 16$: $f^*(1,16) = 8 + (-1) \cdot \underset{(x^*)^2}{(\pm 2)^2} = \underline{\underline{4}}$

iii) $a = 1, b = 20$: $f^*(1,20) = 8 + (-1) \cdot (\pm 2)^2 + 4 \cdot 1/4 = \underline{\underline{5}}$

We use that:

$L = ax^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - b)$

$\frac{\partial L}{\partial a} = x^2 \Rightarrow \frac{df^*}{da} = x^*(a,b)^2$

$\frac{\partial L}{\partial b} = \lambda \Rightarrow \frac{df^*}{db} = \lambda^*(a,b)$

(Exact values: $f^*(2,20) = 4\sqrt{5} \simeq 8.94$)
$f^*(1,16) = 4$
$f^*(1,20) = 2\sqrt{5} \simeq 4.47$