Solutions: Key problems — Lecture 8

1. a) $\min f(x,y) = x^2+y^2$ when $xy=1$

$\mathcal{L} = x^2+y^2-\lambda(xy)$

$\mathcal{L}'_x = 2x - \lambda y = 0$
$\mathcal{L}'_y = 2y - \lambda x = 0$
$xy = 1$

$(x^*,y^*) = (1,1)$ gives $\lambda = \underline{2}$
$\Downarrow$
$\underline{(1,1;2)}$ ord. cond. pt.

$h(x,y) = \mathcal{L}(x,y;2) = x^2+y^2-2xy$

$h'_x = 2x - 2y$
$h'_y = -2x + 2y$

$H(h) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$D_1 = 2 > 0$
$D_2 = 0$
$\Downarrow$
$H(h)$ pos. semidefin. by RRC

$(x^*,y^*) = (1,1)$ is $\underline{\min}$ $\xLeftarrow{\text{SOC}}$ h $\underline{\text{convex}}$ $\Leftarrow$ $H(h)$ pos. semidefin.

$f_{\min} = \underline{\underline{2}}$

b) $\min f(x,y,z) = x^2+y^2+z^2$ when $3x^2+2y^2+2z^2 \geq 12$
$\max -f = -x^2-y^2-z^2$ $\quad\Updownarrow\quad$ $-3x^2-2y^2-2z^2 \leq -12$

$\mathcal{L} = -x^2-y^2-z^2-\lambda(-3x^2-2y^2-2z^2)$

$\mathcal{L}'_x = -2x + 3\lambda\cdot 2x = 0$
$\mathcal{L}'_y = -2y + 2\lambda\cdot 2y = 0$
$\mathcal{L}'_z = -2z + 2\lambda\cdot 2z = 0$
$3x^2+2y^2+2z^2 \geq 12$
$\lambda \geq 0,\ \lambda\cdot(3x^2+2y^2+2z^2-12) = 0$

$(x^*,y^*,z^*) = (2,0,0)$: $-4+4\cdot 3\lambda = 0$
$\lambda = +4/12 = +\underline{1/3}$
$\Downarrow$
$\underline{(2,0,0;1/3)}$ ord. cond. pt.

$h(x,y,z) = -\frac{1}{3}y^2 - \frac{1}{3}z^2$

$H(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$

$\left\{\begin{array}{l} \lambda_1 = 0 \leq 0 \\ \lambda_2 = -2/3 \leq 0 \\ \lambda_3 = -2/3 \leq 0 \end{array}\right.$ neg. semidefin.
$\Downarrow$
$(2,0,0)$ is max for $-f$ $\xLeftarrow{\text{SOC}}$ h $\underline{\text{concave}}$
$\underline{(2,0,0)}$ is min for f
$f_{\min} = \underline{4}$

2. a) $xyz=1$: $\quad \operatorname{rk}(yz \;\; xz \;\; xy)=1$

if $yz=0$, then $y=0$ or $z=0$ $\Rightarrow$ not adm.

That is: this set of eqn. has no Solutions:

$$\left.\begin{array}{l} xyz=1 \\ yz=0 \\ xz=0 \\ xy=0 \end{array}\right\} \operatorname{rk} J<1$$

NDCQ satisfied at all adm. pts. $\Leftarrow$

b) $3x^2+3y^2+8z^2 \geq 1$:

>1 : no cond.

$=1$: $\operatorname{rk}(6x \;\; 6y \;\; 24z)=1$

NDCQ fails if $x=y=z=0$, and this is not adm.

c) $x^3+y^3+z^3=0$: $\quad \operatorname{rk}(3x^2 \;\; 3y^2 \;\; 3z^2)=1$

$x=y=z=0 \Rightarrow (0,0,0)$ is adm. and NDCQ fails

d) $\begin{array}{l} xy-zw=1 \\ x+y+z+w=4 \end{array}$ $\quad \operatorname{rk}\begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix}=2$

NDCQ fails: all 2-minors in J are zero

$\begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix}=0 \quad y-x=0 \quad y=x$

$\begin{vmatrix} x & -w \\ 1 & 1 \end{vmatrix}=0 \quad x+w=0 \quad x=-w$

$\begin{vmatrix} -w & -z \\ 1 & 1 \end{vmatrix}=0 \quad -w+z=0 \quad z=w$

$\Downarrow$

$x=-w, y=-w$
$z=w, w=w$

NDCQ satisfied at all adm. pts. $\Leftarrow$ Not adm since $xy-zw=(-w)^2-w^2=0\neq 1$

3. a) max $f(x,y,z) = 2 - x^2 + 4xy - 5y^2 + 2yz - 2z^2$ when $3x - 2y + z = 10$

$h = f - \lambda\cdot(3x - 2y + z)$

$h'_x = -2x + 4y - 3\lambda = 0$

$h'_y = 4x - 10y + 2z + 2\lambda = 0$

$h'_z = 2y - 4z - \lambda = 0$

$3x - 2y + z = 10$

$$\left(\begin{array}{cccc|c} -2 & 4 & 0 & -3 & 0 \\ 4 & -10 & 2 & 2 & 0 \\ 0 & 2 & -4 & -1 & 0 \\ 3 & -2 & 1 & 0 & 10 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & -18 & -2 & 14 & -40 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & -8 & -2 & 9 & -20 \end{array}\right)$$

$$\leftarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & 0 & -38 & 5 & -40 \\ 0 & 0 & -18 & 5 & -20 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & 0 & -20 & 0 & -20 \\ 0 & 0 & -18 & 5 & -20 \end{array}\right)$$

$x = -2\cdot 9/5 - 1 + 3\cdot(-2/5) + 10 = \underline{21/5}$

$y = (4\cdot 1 - 2/5)/2 = \underline{9/5}$

$z = \underline{1}$

$\lambda = \underline{-2/5}$

Res. cond. pt: $(21/5, 9/5, 1; -2/5)$

Soc: $h = h(x,y,z; -2/5)$

$$H(h) = H(f) = \begin{pmatrix} -2 & 4 & 0 \\ 4 & -10 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

$D_1 = -2$

$D_2 = 4$

$D_3 = -4\cdot 4 - 2\cdot(-4) = -8$

h neg. def $\xRightarrow{\text{Soc}}$ h concave $\Rightarrow (21/5, 9/5, 1)$ is max

$f_{max} = f(21/5, 9/5, 1) = \underline{\underline{0}}$

b) $\max f(x,y,z,w) = xz + yw$ wh $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

$L = xz + yw - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$

FOC: $L'_x = z - \lambda_1 \cdot 2x = 0$ $\quad z = 2\lambda_1 x$

$L'_y = w - \lambda_1 \cdot 2y = 0$ $\quad \Downarrow$ $\quad w = 2\lambda_1 y$

$L'_z = x - \lambda_2 \cdot 8z = 0$ $\quad x = 8\lambda_2 \cdot (2\lambda_1 x) = 16\lambda_1\lambda_2 x$ $\quad \Downarrow$

$L'_w = y - \lambda_2 \cdot 18w = 0$ $\quad y = 18\lambda_2(2\lambda_1 y) = 36\lambda_1\lambda_2 y$

FOC: $x(1-16\lambda_1\lambda_2)=0$ $\quad x=0$ or $\lambda_1\lambda_2 = 1/16$

$y(1-36\lambda_1\lambda_2)=0$ $\quad y=0$ or $\lambda_1\lambda_2 = 1/36$

a) $\underline{x=y=0}$: $z = 2\lambda_1 x = 0$, $w = 2\lambda_2 y = 0$ $\Rightarrow$ $\boxed{\begin{array}{l}(x,y,z,w) = (0,0,0,0) \\ (\lambda_1,\lambda_2) = (0,0) \quad \text{since } C \text{ are } < . \\ f = 0\end{array}}$

b) $\underline{x=0,\ \lambda_1\lambda_2 = 1/36}$: $z = 2\lambda_1 x = 0$, $\lambda_1,\lambda_2 > 0 \Rightarrow C$ are $=$ $\begin{cases} x^2+y^2=1: & y=\pm 1 \\ 4z^2+9w^2=36: & w = \pm 2 \end{cases}$

$w = 2\lambda_1 y$: $w=2,\ y=1,\ \lambda_1 = 1 \Rightarrow \lambda_2 = 1/36$

$\Rightarrow$ $\boxed{\begin{array}{c}(x,y,z,w;\lambda_1,\lambda_2) = (0,\pm 1, 0, \pm 2; 1, 1/36) \\ \text{with } y=w \\ f=2\end{array}}$

c) $y=0,\ \lambda_1\lambda_2 = 1/16$: $w=y=0$, $\lambda_1,\lambda_2>0 \Rightarrow C$ are $=$ $\begin{cases} x^2+y^2=1: & x=\pm 1 \\ 4z^2+9w^2=36: & z=\pm 3\end{cases}$

$z = 2\lambda_1 x \Rightarrow z = x,\ \lambda_1 = 3/2\ \ \lambda_2 = 1/24$

$\boxed{\begin{array}{c}(x,y,z,w;\lambda_1,\lambda_2) = (\pm 1, 0, \pm 3, 0; 3/2, 1/24) \\ \text{with } x=y . \quad f=3\end{array}}$

d) $\left.\begin{array}{l}\lambda_1\lambda_2 = 1/16 \\ \lambda_1\lambda_2 = 1/36\end{array}\right\}$ impossible

$\uparrow$ best candidate for max

$$h = xz + yw - \frac{3}{2}(x^2+y^2) - \frac{1}{24}(4z^2+9w^2)$$

$$H(h) = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/4 \end{pmatrix}$$

since $-\frac{8}{24} = -\frac{1}{3}$, $-\frac{18}{24} = -\frac{3}{4}$

$D_1 = -3$

$D_2 = 9$

$D_3 = 0$

$D_4 = -3/4 \cdot 0 + 1 \cdot \begin{vmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1/3 & 0 \end{vmatrix}$

$= 0 + 0 = 0$

$$\begin{vmatrix} -3 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -1/3 \end{vmatrix} = -3 \cdot 0 = 0$$

$\Delta_1 = -3, -3, -1/3, -3/4 \le 0$

$\Delta_2 = 9, 0, 9/4, 1, 5/4, 1/4 \ge 0$

$\Delta_3 = 0, -15/4, 0, -5/12 \le 0$

$\Delta_4 = 0 \ge 0$

$$\begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3/4 \end{vmatrix} = -3(5/4) = -\frac{15}{4}$$

$$\begin{vmatrix} -3 & 1 & 0 \\ 1 & -1/3 & 0 \\ 0 & 0 & -3/4 \end{vmatrix} = -\frac{3}{4}(0) = 0$$

$$\begin{vmatrix} -3 & 0 & 1 \\ 0 & -1/3 & 0 \\ 1 & 0 & -3/4 \end{vmatrix} = -\frac{1}{3}(5/4) = -\frac{5}{12}$$

By the SOC, it follows that $(\pm 1, 0, \pm 3, 0)$ is max since h is concave

and

$f_{max} = 3$

$(-1, 0, -3, 0)$, $(1, 0, 3, 0)$ are max pts

~~$h = xz + yw - \frac{3}{2}(x^2+y^2) - \frac{1}{24}(4z^2+9w^2)$~~

~~$H(h) = \begin{pmatrix} -3 & 0 & 2 & 0 \\ 0 & -3 & 0 & 2 \\ 2 & 0 & -1/3 & 0 \\ 0 & 2 & 0 & -18/24 \end{pmatrix}$~~ ~~$D_1 = -3$, $D_2 = 9$, $D_3 = -3\cdot(1-4) = 9$~~ ~~h not concave $\Downarrow$ cannot use SOC~~

Alternative method:

EVT: $x^2+y^2 \le 1$, $4z^2+9w^2 \le 36$ $\Rightarrow$ $-1 \le x, y \le 1$, $-3 \le z \le 3$, $-2 \le w \le 2$ — D is bounded

NDCQ: $\left.\begin{matrix} c_1 = \\ c_2 = \end{matrix}\right\}$ $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 8z & 18w \end{pmatrix} = 2$

$\operatorname{rk}(\cdot) < 2$: $x=y=0$ or $z=w=0$ not adm $\Rightarrow$ NDCQ holds

$\left.\begin{matrix} c_1 = \\ c_2 < \end{matrix}\right\}$ $\operatorname{rk}(2x \;\; 2y \;\; 0 \;\; 0) = 1$

$\operatorname{rk}(\cdot) < 1$: $x=y=0$ $\Rightarrow$ not adm. $\Rightarrow$ NDCQ holds

$\left.\begin{matrix} c_1 < \\ c_2 = \end{matrix}\right\}$ $\operatorname{rk}(0 \;\; 0 \;\; 8z \;\; 18w) = 1$

$\operatorname{rk}(\cdot) < 1$: $z=w=0$ $\Rightarrow$ not adm. $\Rightarrow$ NDCQ holds

$\left.\begin{matrix} c_1 < \\ c_2 < \end{matrix}\right\}$ no cond.

NDCQ holds for adm. pts in all 4 cases.

Concl: Best ord. cand. pt is max

$\Rightarrow$ $(1,0,3,0; 3/2, 1/24)$, $(-1,0,-3,0; 3/2, 1/24)$ are max pts

$f_{max} = 3$