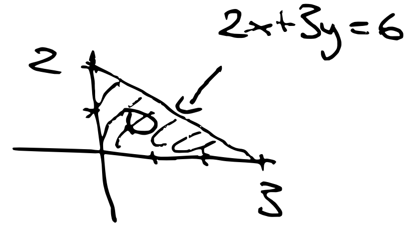
Solutions: Problem Set 7

1.

a) $D: x, y \geq 0$

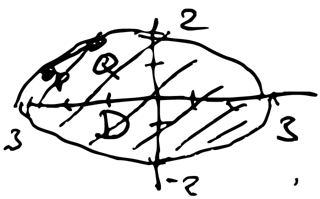
$2x + 3y \leq 6$



compact: yes (bounded since $0 \leq x \leq 3$, $0 \leq y \leq 2$)

convex: yes

b) $D: 4x^2 + 9y^2 \leq 36$

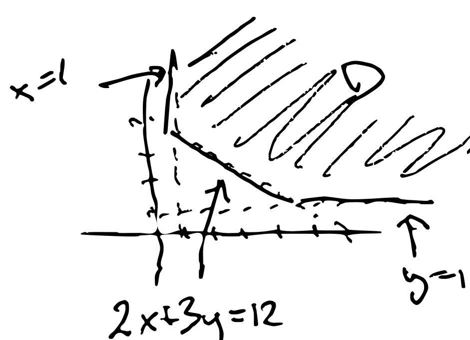


compact: yes

convex: yes

(P, Q in D $\Rightarrow$ line segment [P, Q] in D)

bounded since $-3 \leq x \leq 3$, $-2 \leq y \leq 2$



c) $D: x, y \geq 1$

$2x + 3y \geq 12$

not compact (no upper bound for x or y)

convex

d) $D: xyz \le 1$

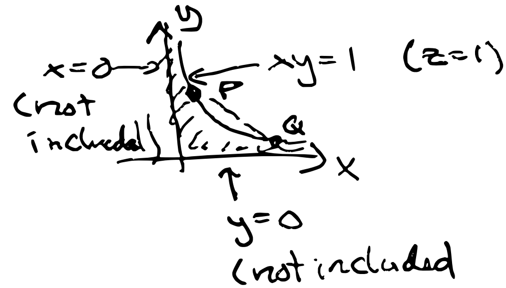
$x, y, z > 0$

<u>not</u> compact

(not closed, not bounded)

<u>not</u> convex

<u>Special case $z=1$:</u>



2.

a) $f = x - y + z$

$f'_x = 1$

$f'_y = -1$

$f'_z = 1$

$$H(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

f <u>convex</u> and <u>concave</u>

since $H(f)$ pos. semidefn. and neg. — " — at all pts.

b) $f = 1 - e^u$, $u = x - y + z$

$f'_x = -e^u \cdot 1 = -e^u$

$f'_y = -e^u \cdot (-1) = e^u$

$f'_z = -e^u \cdot (1) = -e^u$

$$H(f) = \begin{pmatrix} -e^u & e^u & -e^u \\ e^u & -e^u & e^u \\ -e^u & e^u & -e^u \end{pmatrix}$$

$$\left.\begin{array}{l} rk\ H(t) = 1 \\ D_1 = -e^u < 0 \end{array}\right\} \overset{RRC}{\Longrightarrow} H(t) \text{ neg. semid. at all pts}$$

$\Downarrow$

f <u>concave</u>

c) $f = u^4$, $u = x+y+z+w$

$$f'_x = 4u^3 \cdot 1$$
$$\vdots \qquad \vdots$$
$$f'_w = 4u^3 \cdot 1$$

$$H(t) = \begin{pmatrix} 12u^2 & 12u^2 & 12u^2 & 12u^2 \\ \vdots & \vdots & & \vdots \\ 12u^2 & 12u^2 & 12u^2 & 12u^2 \end{pmatrix}$$

$rk\ H(t) = 1$, $u \neq 0$

$rk\ H(t) = 0$, $u = 0$

$u = 0$: $H(t) = 0$ pos. semidef.

$u \neq 0$: $rk\ H(t) = 1$

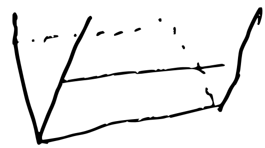
$D_1 = 12u^2 > 0$

$\Downarrow$ RRC

$H(t)$ pos. semid. at all pts

f <u>convex</u>

d) $f = |x-y|$



graph of f

All pts over the graph of f ("inside") = convex set

f convex $\Longleftarrow$ convex set

3.

a) max $f = x+2y+3z$ under $2x^2+y^2+3z^2=9$

$$L = x+2y+3z-\lambda(2x^2+y^2+2z^2)$$

FOC:
$$\begin{cases} L'_x = 1-\lambda\cdot 4x = 0 & \Rightarrow x = \frac{1}{4\lambda} \quad (\lambda \neq 0) \\ L'_y = 2-\lambda\cdot 2y = 0 & y = \frac{2}{2\lambda} = \frac{4}{4\lambda} \\ L'_z = 3-\lambda\cdot 4z = 0 & z = \frac{3}{4\lambda} \end{cases}$$

C: $2x^2+y^2+2z^2=9$

$$2\cdot\left(\frac{1}{4\lambda}\right)^2+\left(\frac{4}{4\lambda}\right)^2+2\cdot\left(\frac{3}{4\lambda}\right)^2=9 \quad | \cdot (4\lambda)^2$$

$$2\cdot 1^2+4^2+2\cdot 3^2 = 9\cdot(4\lambda)^2$$

$$36 = 9\cdot(4\lambda)^2 \quad \Big|\cdot 1/9$$

$$4 = (4\lambda)^2$$

$$4\lambda = \pm\sqrt{4} = \pm 2$$

$$\lambda = \pm\frac{2}{4} = \pm\frac{1}{2}$$

Cond. pts:

$\lambda = \frac{1}{2}$: $(x,y,z;\lambda) = (1/2, 2, 3/2; 1/2)$ $f=9$

$\lambda = -\frac{1}{2}$: — " — $= (-1/2, -2, -3/2; -1/2)$ $f=-9$

D compact $\overset{EVT}{\Longrightarrow}$ there is a $\Longrightarrow$ $\underline{\underline{f_{max} = 9}}$

(bounded) max NDCQ at $(1/2, 2, 3/2)$

$$2x^2 + y^2 + 2z^2 = 9$$

$$\Downarrow$$

$$\begin{aligned} x^2 &\le 9/2 \\ y^2 &\le 9 \\ z^2 &\le 9/2 \end{aligned} \quad \left(\begin{aligned} -\sqrt{9/2} &\le x \le \sqrt{9/2} \\ -3 &\le y \le 3 \\ -\sqrt{9/2} &\le z \le \sqrt{9/2} \end{aligned} \right)$$

b) max/min $f = x^4 + y^4 + z^4$ when $2x^2 + y^2 + 2z^2 = 9$

$$L = x^4 + y^4 + z^4 - \lambda(2x^2 + y^2 + 2z^2)$$

FOC:
$$\begin{cases} L'_x = 4x^3 - \lambda \cdot 4x = 0 & 4x(x^2 - \lambda) = 0 \\ L'_y = 4y^3 - \lambda \cdot 2y = 0 & 4y(y^2 - \lambda/2) = 0 \\ L'_z = 4z^3 - \lambda \cdot 4z = 0 & 4z(z^2 - \lambda) = 0 \end{cases}$$

C: $2x^2 + y^2 + 2z^2 = 9$

FOC:
$$\begin{aligned} x &= 0 \text{ or } x^2 = \lambda \\ y &= 0 \text{ or } y^2 = \lambda/2 \\ z &= 0 \text{ or } z^2 = \lambda \end{aligned}$$

$\Downarrow$

Cases:

a) $x = y = z = 0$: $2x^2 + y^2 + 2z^2 = 0 \neq 9$
no solutions

b) $x = y = 0, \lambda = z^2$: $2\lambda = 9 \Rightarrow \lambda = 9/2 = z^2$

$(x, y, z; \lambda) = (0, 0, \pm\sqrt{9/2}; 9/2)$ $f = 81/4$

c) $x = z = 0, y^2 = \lambda/2$: $\lambda/2 = 9 \Rightarrow \lambda = 18, y = \pm 3$

$(x, y, z; \lambda) = (0, \pm 3, 0; 18)$ $f = 81$

d) $y = z = 0, x^2 = \lambda$: $2\lambda = 9$, $\lambda = 9/2 = x^2$

$(x, y, z; \lambda) = (\pm\sqrt{9/2}, 0, 0; 9/2)$ $f = 81/4$

e) $x=0,\ y^2=\lambda/2,\ z^2=\lambda$: $\quad \lambda/2+2\lambda=9$

$$5\lambda=18$$

$$\lambda=18/5$$

$$y^2=9/5 \quad z^2=18/5$$

$$(x,y,z;\lambda)=\left(0,\pm\sqrt{9/5},\pm\sqrt{18/5};\ 18/5\right)$$

$$f=(9/5)^2+(18/5)^2$$

$$=\frac{9^2+18^2}{5^2}=\frac{5\cdot 9^2}{5^2}=\underline{81/5}$$

f) $x^2=\lambda,\ y=0,\ z^2=\lambda$: $\quad 2\lambda+2\lambda=9 \quad \lambda=9/4$

$$x^2=z^2=9/4$$

$$(x,y,z;\lambda)=\left(\pm 3/2, 0, \pm 3/2;\ 9/4\right) \quad f=\frac{81}{16}\cdot 2=\underline{81/8}$$

g) $x^2=\lambda,\ y^2=\lambda/2,\ z=0$: (as in case e)

$$(x,y,z;\lambda)=\left(\pm\sqrt{18/5},\pm\sqrt{9/5},0;\ 18/5\right) \quad f=\underline{81/5}$$

h) $x^2=\lambda,\ y^2=\lambda/2,\ z^2=\lambda$: $\quad 2\lambda+\lambda/2+2\lambda=9$

$$9\lambda=18 \Rightarrow \underline{\lambda=2}$$

$$x^2=2,\ y^2=1,\ z^2=2$$

$$(x,y,z;\lambda)=\left(\pm\sqrt{2},\pm 1,\pm\sqrt{2};\ 2\right) \quad f=\underline{9}$$

$2x^2+y^2+2z^2=9$ is compact $\xRightarrow{\text{EVT}}$ there is max/min $\xRightarrow{\text{NDCQ}}$

$f_{max} = \underline{\underline{81}}$ at $(0, \pm\sqrt{3}, 0)$

$f_{min} = \underline{\underline{9}}$ at $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2})$

4.

a) max $f = x-2y+z$ when $x^2+y^2+z^2 \leq 3$ ← KT prob. in <u>std</u> form

$L = x-2y+z-\lambda(x^2+y^2+z^2)$

<u>FOC</u>:

$L'_x = 1-\lambda\cdot 2x = 0 \Rightarrow x = 1/2\lambda$ $\quad(\lambda \neq 0)$

$L'_y = -2-\lambda\cdot 2y = 0 \qquad y = -2/2\lambda$

$L'_z = 1-\lambda\cdot 2z = 0 \qquad z = 1/2\lambda$

<u>C</u>: $x^2+y^2+z^2 \leq 3$ $\quad$ <u>CSC</u>: $\lambda \geq 0$, $\lambda(x^2+y^2+z^2-3)=0$

$\lambda \neq 0 \xRightarrow{\text{CSC}} x^2+y^2+z^2=3$: $\left(\frac{1}{2\lambda}\right)^2+\left(\frac{-2}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=3$

$1^2+(-2)^2+1^2 = 3\cdot(2\lambda)^2$

$6/3 = (2\lambda)^2$

$2\lambda = \pm\sqrt{2}$

$\lambda = \pm\sqrt{2}/2$

$$\boxed{(x,y,z;\lambda) = (\pm 1/\sqrt{2}, \mp 2/\sqrt{2}, \pm 1/\sqrt{2}; \pm\sqrt{2}/2)}$$

Cond. pts: $(x,y,z;\lambda) = (1/\sqrt{2}, -2/\sqrt{2}, 1/\sqrt{2}; \sqrt{2}/2)$

with $f = 6/\sqrt{2}$

(since $\lambda > 0$ gives $\lambda = \sqrt{2}/2$)

D is bounded $\overset{\text{EVT}}{\Longrightarrow}$ there is a max $\overset{\text{NDCQ}}{\Longrightarrow}$

$x^2+y^2+z^2 \le 3$

$$\left(\begin{array}{l} -\sqrt{3} \le x \le \sqrt{3} \\ -\sqrt{3} \le y \le \sqrt{3} \\ -\sqrt{3} \le z \le \sqrt{3} \end{array}\right)$$

$f_{max} = \frac{6}{\sqrt{2}} = \frac{3\cdot 2}{\sqrt{2}} = \underline{3\sqrt{2}}$

at $(1/\sqrt{2}, -2/\sqrt{2}, 1/\sqrt{2})$

b) max $f = \ln(xyz)$ when $2x^2+y^2+2z^2 \le 6$

Note: KT problem in std form

f defined when $xyz > 0$

Solve as KT problem, check $xyz > 0$

$$L = \ln(xyz) - \lambda(2x^2 + y^2 + 2z^2)$$

<u>FOC</u>: $L'_x = \frac{1}{xyz} \cdot yz - \lambda \cdot 4x = 0 \qquad \frac{1}{x} - 4\lambda x = 0$

$L'_y = \frac{1}{xyz} \cdot xz - \lambda \cdot 2y = 0 \qquad \frac{1}{y} - 2\lambda y = 0$

$L'_z = \frac{1}{xyz} \cdot xy - \lambda \cdot 4z = 0 \qquad \frac{1}{z} - 4\lambda z = 0$

$(\lambda \neq 0)$

<u>C</u>: $2x^2 + y^2 + z^2 \leq 6$ <u>CSC</u>: $\lambda \geq 0$

$\lambda(2x^2 + y^2 + 2z^2 - 6) = 0$

$\lambda \neq 0 \Rightarrow \underline{2x^2 + y^2 + 2z^2 = 6}$

CSC

<u>FOC</u>: $1 - 4\lambda x^2 = 0 \Rightarrow x^2 = 1/4\lambda$

$1 - 2\lambda y^2 = 0 \qquad y^2 = 1/2\lambda$

$1 - 4\lambda z^2 = 0 \qquad z^2 = 1/4\lambda$

<u>C</u>: $2x^2 + y^2 + 2z^2 = 6$

$2\left(\frac{1}{4\lambda}\right) + \left(\frac{1}{2\lambda}\right) + 2\left(\frac{1}{4\lambda}\right) = 6 \qquad | \cdot 4\lambda$

$2 + 2 + 2 = 6 \cdot 4\lambda$

$4\lambda = 1 \qquad \lambda = 1/4 \Rightarrow x^2 = 1 \quad y^2 = 2 \quad z^2 = 1$

$\lambda > 0$

$x = \pm 1, \ y = \pm\sqrt{2}, \ z = \pm 1 \qquad \lambda = 1/4$

Cand. pts:

$$(x, y, z; \lambda) = (1, \sqrt{2}, 1; 1/4) \qquad f = \ln(\sqrt{2})$$
$$(-1, -\sqrt{2}, 1; 1/4) \qquad \text{"}$$
$$(-1, \sqrt{2}, -1; 1/4) \qquad \text{"}$$
$$(1, -\sqrt{2}, -1; 1/4) \qquad \text{"}$$

four of eight pts have $\boxed{xyz > 0}$

If f had been defined everywhere inside $2x^2 + y^2 + 2z^2 \le 6$, then by EVT we would have a max, a $f_{max} = \ln(\sqrt{2}) = \ln(2^{1/2})$
$= \underline{\underline{\frac{1}{2}\ln 2}}$ at the four points given above.

In fact, $D: \begin{matrix} 2x^2 + y^2 + 2z^2 \le 6 \\ xyz > 0 \end{matrix}$ is <u>not</u> <u>closed</u>, so we must check points close to $xyz = 0$ (boundary)

Since $\ln(u) \to -\infty$ when $u \to 0^+$, this will not give higher f-values than $\frac{1}{2}\ln 2$.

Therefore, we can conclude:

$$f_{max} = \underline{\underline{\frac{1}{2}\ln 2}}$$