Solutions: Problem Set 5

1. a) $A = \begin{pmatrix} 0.30 & 0.15 \\ 0.70 & 0.85 \end{pmatrix}$ regular Markov chain

$\lambda = 1$: $\begin{pmatrix} -0.70 & 0.15 \\ 0.70 & -0.15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ one free var.

Alt 1 $\quad -0.70x + 0.15y = 0$

y free

$$x = \frac{-0.15y}{-0.70} = \frac{15}{70}y = \frac{3}{14}y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/14 \cdot y \\ y \end{pmatrix} = y \begin{pmatrix} 3/14 \\ 1 \end{pmatrix}$$

$x + y = 1$: $\quad \frac{3}{14}y + y = 1 \quad \frac{17}{14}y = 1 \quad y = \frac{14}{17}$

$$x = \frac{3}{14} \cdot \frac{14}{17} = \underline{\frac{3}{17}} \qquad y = \underline{\frac{14}{17}}$$

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/17 \\ 14/17 \end{pmatrix} \quad \text{eq. state}$$

Alt 2: $\quad -0.70x + 0.15y = 0$

see → that this is one sol.

$x = 15 \quad y = 70 \Rightarrow \underline{v} = \begin{pmatrix} 15/85 \\ 70/85 \end{pmatrix} = \begin{pmatrix} 3/17 \\ 14/17 \end{pmatrix}$

$(15 + 70 = 85)$

b) $A = \begin{pmatrix} 0.86 & 0.42 \\ 0.14 & 0.58 \end{pmatrix}$ regular Markov ch.

$\underline{\lambda=1}$: $\begin{pmatrix} -0.14 & 0.42 \\ 0.14 & -0.42 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left.\begin{array}{l} x = 42 \quad y = 14 \\ x+y = 56 \end{array}\right\}$ $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 42/56 \\ 14/56 \end{pmatrix}$

$= \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$

c) $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$ regular

$\lambda=1$: $\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \begin{array}{l} \cdot 20 \\ \cdot 20 \\ \cdot 20 \end{array}$

$\to \begin{pmatrix} 1 & 1.6 & -6 \\ 4 & -2 & 4 \\ -5 & 0.4 & 2 \end{pmatrix} \begin{array}{l} -4 \\ 5 \end{array} \to \begin{pmatrix} 1 & 1.6 & -6 \\ 0 & -8.4 & 28 \\ 0 & 8.4 & -28 \end{pmatrix}$

$-8.4y + 28z = 0 \Rightarrow y = \frac{28}{8.4}z = \frac{140}{42}z = \frac{10}{3}z$

$x + 1.6y - 6z = 0 \Rightarrow x = 6z - 1.6 \cdot \left(\frac{10}{3}z\right)$

$= \frac{18-16}{3}z = \frac{2}{3}z$

$x+y+z=1$

$\frac{2}{3}z + \frac{10}{3}z + z = 1$

$\Rightarrow 2z + 10z + 3z = 3 \Rightarrow z = \underline{3/15}$

$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$

d) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

not clear if this is a regular Markov chain

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\Downarrow$

$A^n = A$ (n odd) $\qquad A^n = I$ (n even)

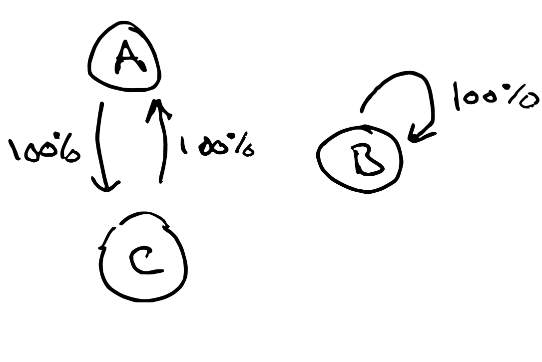
<u>Not regular:</u> $\qquad \underline{v_0} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$A^n \cdot \underline{v_0} = A^n \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, & n \text{ odd} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, & n \text{ even} \end{cases}$$

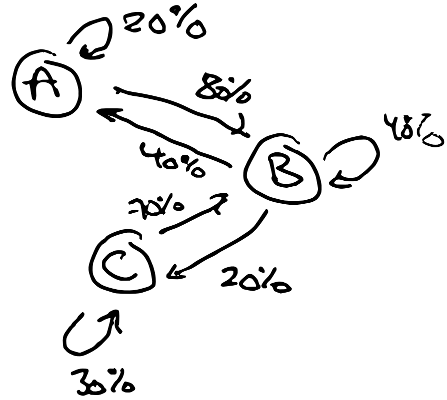
no equilibrium state if $a \neq c$

eq. state $\underline{u} = \underline{\begin{pmatrix} a \\ b \\ a \end{pmatrix}}$ if $a = c$

<u>Note:</u>



e) $A = \begin{pmatrix} 0.2 & 0.4 & 0 \\ 0.8 & 0.4 & 0.7 \\ 0 & 0.2 & 0.3 \end{pmatrix}$



$\underline{\lambda = 1}$: $\begin{pmatrix} -0.8 & 0.4 & 0 \\ 0.8 & -0.6 & 0.7 \\ 0 & 0.2 & -0.7 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} -0.8 & 0.4 & 0 \\ 0 & -0.2 & 0.7 \\ 0 & 0.2 & -0.7 \end{pmatrix}$

$0.2y - 0.7z = 0$

$\underline{y = \frac{7}{2}z}$

$-0.8x + 0.4y = 0$

$-0.8x = -0.4y = -0.4\left(\frac{7}{2}z\right) = -1.4z$

$x = \frac{-1.4}{-0.8}z = \frac{14}{8}z = \frac{7}{4}z$

$x + y + z = 1$

$\frac{7}{4}z + \frac{7}{2}z + z = 1 \quad \cdot 4$

$(7 + 14 + 4)z = 4$

$z = 4/25$

$\underline{v} = \begin{pmatrix} 7/25 \\ 14/25 \\ 4/25 \end{pmatrix}$ eq. state

Note: can get from any node to any other node in **two** steps

$\Downarrow$

regular Markov ch.

2. a) $A = \begin{pmatrix} 7 & 4 \\ 4 & 3 \end{pmatrix}$ $D_1 = 7$, $D_2 = 5$ pos. defn

b) $A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$ $D_1 = -1$, $D_2 = 2$ neg. defn.

c) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ $D_1 = 4$, $D_2 = 20$, $D_3 = 5 \cdot 15 = 75$ pos. defn.

d) $A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & 7 & 0 \\ -5 & 0 & 35 \end{pmatrix}$ $D_1 = 2$, $D_2 = 5$, $D_3 = -5 \cdot (35) + 35 \cdot 5 = 0$

AUHI: $A_1 = 2, 7, 35$; $A_2 = 5, 245, 45$; $A_3 = 0$ $\}$ pos. semidefn.

AUZ: $\operatorname{rk} A = 2 \xrightarrow[\text{RRC}]{}$ pos. semidefn.

e) $A = \begin{pmatrix} -1 & -2 & -2 \\ -2 & -4 & -4 \\ -2 & -4 & -2 \end{pmatrix}$ $D_1 = -1$, $D_2 = 0$, $D_3 = \dots$ $A_1 = -1, -4, -2$; $A_2 = 0, -8, \dots$

indefinite

3.

a) $A = \begin{pmatrix} 1 & -4 \\ -4 & 3 \end{pmatrix}$

$D_1 = 1$
$D_2 = -13$

indefinite

b) $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$D_1 = 2$
$D_2 = 6$
$D_3 = 3 \cdot 1 = 3$

pos. defin

c) $A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 8 \end{pmatrix}$

$D_1 = 3$
$D_2 = 5$
$D_3 = 3 \cdot 20 - 2 \cdot 20 - 2 \cdot 10$
$= 0$

$\Delta_1 = 3, 3, 8$
$\Delta_2 = 5, 20, 20$
$\Delta_3 = 0$
(or RRC)

pos. semidefin

d) $A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$

$D_1 = 0$
$D_2 = 0$
$D_3 = 0$
$D_4 = -1/2 \cdot (-1/2)\,(1/2)^2 = 1/16$
$\Delta_2^{23,23} = 0 - 1/4 = -1/4 < 0$

indefinite

4.

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 1$
$D_3 = 0$
$D_4 = 0$

Alt 1: $\Delta_1 = 1, 1, 1, 1$
$\Delta_2 = 1, 1, 0, 0, 1, 1$
$\Delta_3 = 0, 0, 0, 0$
$\Delta_4 = 0$

pos. semidefin.

Alt 2: $\operatorname{rk} A = 2 \xRightarrow{\text{REC}}$ pos. semidefin.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$