Solutions: Problem Set 2

$$\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \underline{v_4} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \quad \underline{v_5} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

1.

a)

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix}} \\ \begin{matrix} 1 & -1 \end{matrix} \end{pmatrix}$$

$M_{12,12} = 4 - 2 = 2 \neq 0$

$\Rightarrow \{\underline{v_1}, \underline{v_2}\}$ lin. independent

b)

$$\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -2 \\ 1 & -1 & 5 \end{vmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_3} \end{matrix} \end{array} = 1 \cdot (20-2) - 1(10+2) + 1(-2-4)$$

$$= 18 - 12 - 6 = \underline{0}$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ not linearly independent

c)

$$\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{vmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_5} \end{matrix} \end{array} = 1 \cdot (12-1) - 1(6+1) + 1 \cdot (-2-4)$$

$$= 11 - 7 - 6 = -2 \neq 0$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_5}\}$ lin. independent

d) $\begin{vmatrix} 1 & 1 & -1 \\ 4 & -2 & 3 \\ -1 & 5 & -7 \end{vmatrix} = 1\cdot(14-15) - 1(-28+3) - 1(20-2)$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{v_2} & \underline{v_3} & \underline{v_4} \end{array}$$

$= -1 + 25 - 18 = 6 \neq 0$

$\{\underline{v_2}, \underline{v_3}, \underline{v_4}\}$ lin independent

e) $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -2 & 3 \\ 1 & -1 & 5 & -7 \end{pmatrix}$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} \end{array}$$

is 3×4, rk is at most $\underline{3}$

so

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ **not** lin. independent

2. a) dim $\underline{V=2}$, Base $\underline{\{v_1, v_2\}}$ since $\underline{v_1}, \underline{v_2}$ lin. indep.

b) dim $\underline{V=2}$, Base $\underline{\{v_1, v_2\}}$

since dim $V < 3$ and $\{\underline{v_1}, \underline{v_2}\}$ lin independent

Alt: $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -2 \\ 1 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{array}{ccc} \uparrow & \uparrow & \\ \underline{v_1} & \underline{v_2} & \end{array}$$

c) dim $V = \underline{3}$, Base

$\underline{\{\underline{v_1}, \underline{v_2}, v_5\}}$ since $\underline{v_1}, \underline{v_2}, \underline{v_3}$ lin. indep.

d) $\dim V = \underline{3}$ Base $\underline{\{\underline{u}_2, \underline{u}_3, \underline{u}_4\}}$

since $\dim V < 4$ and $\{\underline{u}_2, \underline{u}_3, \underline{u}_4\}$ lin. indep.

Alt:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -2 & 3 \\ 1 & -1 & 5 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \\ 0 & -2 & 4 & -6 \end{pmatrix} \to \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ \underline{v}_1 & \underline{v}_2 & & \underline{v}_4 \end{matrix} \end{array}$$

Another base is $\underline{\{\underline{u}_1, \underline{u}_2, \underline{u}_4\}}$

3. $A = (\underline{v}_1 | \underline{u}_2 | \underline{v}_3 | \underline{u}_4 | \underline{u}_5)$

a) Null(A): $\boxed{A\underline{x} = 0} \leftarrow$ solutions of

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 2 & 4 & -2 & 3 & -1 \\ 1 & -1 & 5 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 5 & -3 \\ 0 & -2 & 4 & -6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 5 & -3 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$ echelon form

$x_1 = -(2x_3 + 4x_5) - x_3 + (-x_5) - x_5 = \underline{-3x_3 - 6x_5}$

$2x_2 = 4x_3 - 5(-x_5) + 3x_5 \Rightarrow x_2 = \underline{2x_3 + 4x_5}$

$-x_4 - x_5 = 0 \Rightarrow x_4 = \underline{-x_5}$

x_3, x_5: free
x_1, x_2, x_4: basic

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 6x_5 \\ 2x_3 + 4x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \underbrace{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\underline{w}_1} + x_5 \underbrace{\begin{pmatrix} -6 \\ 4 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\underline{w}_2}$$

$Null(A) = \underline{\underline{span(\underline{w}_1, \underline{w}_2)}}$

Base: $\underline{\underline{\{\underline{w}_1, \underline{w}_2\}}}$ (lin. independent)

b) $rk A = \underline{3} \Rightarrow \dim Col(A) = \underline{3}$

Since there are pivots in col. 1, 2, 4 $\Rightarrow$ Base: $\underline{\underline{\{\underline{v}_1, \underline{v}_2, \underline{v}_4\}}}$

$Col(A) \subseteq \mathbb{R}^3$ and $\dim Col(A) = 3$

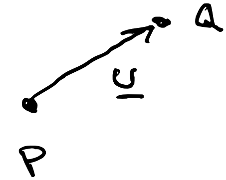
$$\Downarrow$$

$$\underline{\underline{Col(A) = \mathbb{R}^3}}$$

4.

$P = (1,3,2,5)$

$Q = (-2,4,5,1)$



$\vec{PQ} = \underline{u} = (-2-1, 4-3, 5-2, 1-5)$

$= (-3, 1, 3, -4)$

$(x, y, z, w) = P + t \cdot PQ \qquad (0 \le t \le 1)$

$= (1,3,2,5) + t(-3,1,3,-4)$

$= \underline{\underline{(1-3t, 3+t, 2+3t, 5-4t)}}$

or $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 3 \\ -4 \end{pmatrix}$

Intersection:

$w = 9$

$5 - 4t = 9$

$-4t = 4 \Rightarrow \underline{t = -1}$

$t = -1: \ \underline{\underline{(4, 2, -1, 9)}}$

<u>5</u>

$$A = \underbrace{\begin{pmatrix} \cdot & \cdots & \cdot \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & \cdot \end{pmatrix}}_{7} \; 5 \qquad \mathrm{rk}(A) \le 5$$

$\dim \mathrm{Col}(A) = \mathrm{rk}(A)$ $\leftarrow$ # pivots

$\dim \mathrm{Null}(A) = 7 - \mathrm{rk}(A)$ $\leftarrow$ # free var's

$$\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A) = \mathrm{rk}(A) + 7 - \mathrm{rk}(A)$$
$$= \underline{\underline{7}}$$