Solutions: Problem Set 12

1.

a) $$\underline{y}' = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix} \underline{y} \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Eigenvalues / eigenvectors of $A = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}$:

$$\begin{vmatrix} 2-\lambda & -5 \\ -5 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 21 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(-21)}}{2} = \frac{4 \pm \sqrt{100}}{2}$$

$$\underline{\lambda_1 = 7}, \quad \underline{\lambda_2 = -3}$$

$\underline{E_7}$: $\begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \rightarrow E_7 = \text{span}(\underline{v}_1), \ \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{E_{-3}}$: $\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \rightarrow E_{-3} = \text{span}(\underline{v}_2), \ \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution of the system of diff. eqn:

$$\underline{y} = c_1 \cdot \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} = \underline{\underline{c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}}}$$

b) $\underline{y}' = A\underline{y}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$

<u>Eigenvalues / eigenvectors</u> of A:

$$\begin{vmatrix} -\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9 - 4(-4)}}{2} = \frac{3 \pm 5}{2}$$

$$\lambda_1 = \underline{4}, \quad \lambda_2 = \underline{-1}$$

$\underline{E_4}$: $\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix} \rightarrow E_4 = \mathrm{span}(\underline{v_1})$, $\underline{v_1} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\underline{E_{-1}}$: $\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \rightarrow E_{-1} = \mathrm{span}(\underline{v_2})$, $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution of the system of diff. eqn:

$$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} = \underline{\underline{c_1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}}}$$

$\underline{2.}$ $\quad y' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \underline{y}$

<u>Eigenvalues / eigenvectors</u> of A:

$$\begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)\cdot(\lambda^2+10\lambda+24) = 0$$

$\underline{\lambda_1 = -3}$ $\qquad \lambda = \dfrac{-10 \pm \sqrt{100-4\cdot 24}}{2} = \dfrac{-10 \pm 2}{2}$

$$\lambda_2 = \underline{-4}, \quad \lambda_3 = \underline{-6}$$

$\underline{E_{-3}}$: $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \to E_{-3} = \text{span}(\underline{v_1}),\ \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{E_{-4}}$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to E_{-4} = \text{span}(\underline{v_2}),\ \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{E_{-6}}$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to E_{-6} = \text{span}(\underline{v_3}),\ \underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

General solution of the system of diff. eqn:

$$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}$$

$$= c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-4t} + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-6t}$$

3. $$\underline{y}_{t+1} = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \underline{y}_t$$

System of linear difference eqn's, same eigenvalues/eigenvectors as in Problem 2.

<u>General solution:</u>

$$\underline{y}_t = c_1 \cdot \underline{v_1} \cdot \lambda_1^t + c_2 \underline{v_2} \lambda_2^t + c_3 \underline{v_3} \lambda_3^t$$

$$= c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot (-3)^t + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot (-4)^t + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot (-6)^t$$