

Solutions: Key problems lecture 10

1. a) $y' = 3t^2 + 2$
 $y = \int 3t^2 + 2 dt = \underline{t^3 + 2t + C}$

b) $ty' = 1$
 $y' = \frac{1}{t}$
 $y = \int \frac{1}{t} dt = \underline{\ln|t| + C}$

c) $y' = t\sqrt{t^2+1}$
 $y = \int t\sqrt{t^2+1} dt = \int \cancel{t} \sqrt{u} \cdot \frac{du}{2\cancel{t}} = \int \frac{1}{2} u^{1/2} du$
 $u = (t^2+1)$
 $du = 2t dt$
 $= \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) + C = \frac{1}{3} u^{3/2} + C = \underline{\underline{\frac{1}{3} (t^2+1) \sqrt{t^2+1} + C}}$

2. a) $y' = 5y$
 $\frac{1}{y} y' = 5$
 $\int \frac{1}{y} y' dt = \int 5 dt$
 $\frac{dy}{y}$

$\int \frac{1}{y} dy = \int 5 dt$
 $\ln|y| = 5t + C$
 $|y| = e^{5t+C} = e^{5t} \cdot e^C$
 $y = \pm e^C e^{5t} = Ke^{5t}$
 $y = \underline{\underline{Ke^{5t}}}$

have combined C_1, C_2 in the two integrals

b) $y' = y^2 t$
 $\frac{1}{y^2} y' = t$
 $\int \frac{1}{y^2} y' dt = \int t dt$

$\int y^{-2} dy = \int t dt$
 $-\frac{1}{y} = \frac{1}{2} t^2 + C$
 $-\frac{1}{y} = \frac{1}{2} t^2 + C$

$\frac{1}{y} = -\frac{1}{2} t^2 - C$
 $y = \frac{1}{-\frac{1}{2} t^2 - C} \quad (-2)$
 $= \underline{\underline{\frac{-2}{t^2 + 2C}}}$

$$c) y' = 5y(1 - y/10) = 5y \cdot \frac{1}{10} \cdot (10 - y)$$

$$\frac{1}{y(10-y)} y' = 5 \cdot \frac{1}{10} = \frac{1}{2}$$

$$\int \frac{1}{y(10-y)} y' dt = \int \frac{1}{2} dt$$

$$(10-y)' = -1$$

$$\int \frac{1}{y(10-y)} dy = \frac{1}{2} t + C$$

$$\int \frac{1/10}{y} + \frac{1/10}{10-y} dy = \frac{1}{2} t + C$$

$$\frac{1}{10} \ln |y| - \frac{1}{10} \ln |10-y| = \frac{1}{2} t + C$$

$$\ln |y| - \ln |10-y| = 10 \left(\frac{1}{2} t + C \right)$$

$$\ln \left| \frac{y}{10-y} \right| = \frac{5}{2} t + 10C$$

$$\left| \frac{y}{10-y} \right| = e^{\frac{5}{2} t + 10C}$$

$$\frac{y}{10-y} = \pm e^{10C} e^{\frac{5}{2} t} = K e^{\frac{5}{2} t}$$

$$y = K e^{\frac{5}{2} t} \cdot (10-y)$$

$$y = 10K e^{\frac{5}{2} t} - K e^{\frac{5}{2} t} \cdot y$$

$$y + K e^{\frac{5}{2} t} y = 10K e^{\frac{5}{2} t}$$

$$y = \frac{10K e^{\frac{5}{2} t}}{1 + K e^{\frac{5}{2} t}}$$

$$\frac{1}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad | y(10-y)$$

$$1 = A \cdot (10-y) + B \cdot y$$

$$= 10A - Ay + By$$

$$1 = 10A + (B-A)y$$

≡ compare coeff.

$$B-A=0 \quad (\text{no } y\text{-term on LHS})$$

$$10A=1 \quad (\text{const. term})$$

∴

$$\underline{A = \frac{1}{10}} \quad B = A = \underline{\frac{1}{10}}$$

$$\frac{1}{y(10-y)} = \frac{1/10}{y} + \frac{1/10}{10-y}$$

3. a) $y' + 3y = 6$

Alt A: integrating factor $y' + a(t)y = b(t)$
 $a(t) = 3 \Rightarrow \int a(t)dt = 3t + C = u = \underline{\underline{e^{3t}}}$

$(y' + 3y)e^{3t} = 6e^{3t}$
 $(y \cdot e^{3t})' = 6e^{3t}$ follows from choice of u
(theory)

$y e^{3t} = \int 6e^{3t} dt = 6 \cdot \frac{1}{3} e^{3t} + C = 2e^{3t} + C$

$y = \frac{2e^{3t} + C}{e^{3t}} = 2 + \frac{C}{e^{3t}} = \underline{\underline{2 + C \cdot e^{-3t}}}$

Alt B: superposition principle

$y = y_h + y_p = \underline{\underline{C \cdot e^{-3t} + 2}}$

y_h : $y' + 3y = 0$

$r + 3 = 0$

$r = -3$

$\Rightarrow y_h = C \cdot e^{-3t}$

y_p : $y' + 3y = 6$

Try: $y = A$ (const.)
 $y' = 0$
 \Downarrow

$y' + 3y = 6$

$0 + 3A = 6$

$\underline{\underline{A = 2}} \Rightarrow y_p = 2$

$$b) y' - 2ty = 4t$$

$$u = e^{\int -2t dt} = e^{-t^2} = \underline{e^{-t^2}}$$

$$(ye^{-t^2})' = 4te^{-t^2}$$

$$ye^{-t^2} = \int 4te^{-t^2} dt = \int 4te^u \frac{du}{-2t} = -2 \int e^u du$$

$$\boxed{u = -t^2}$$

$$\boxed{du = -2t dt}$$

$$ye^{-t^2} = -2e^u + C = -2e^{-t^2} + C$$

$$y = \frac{-2e^{-t^2} + C}{e^{-t^2}} = -2 + \frac{C}{e^{-t^2}} = \underline{\underline{-2 + Ce^{t^2}}}$$

$$c) y' + 2y = e^t$$

Alt A: $u = e^{\int 2 dt} = e^{2t}$

$$(ye^{2t})' = e^t \cdot e^{2t} = e^{3t}$$

$$ye^{2t} = \int e^{3t} dt = \frac{1}{3}e^{3t} + C$$

$$y = \frac{1}{3} \frac{e^{3t}}{e^{2t}} + \frac{C}{e^{2t}} = \underline{\underline{\frac{1}{3}e^t + Ce^{-2t}}}$$

Alt B:

$$y = y_h + y_p = \underline{\underline{Ce^{-2t} + \frac{1}{3}e^t}}$$

y_h:

$$y' + 2y = 0$$

$$r + 2 = 0$$

$$r = -2$$

$$\Rightarrow y_h = Ce^{-2t}$$

y_p:

$$y' + 2y = e^t$$

Try: $y = Ae^t$ (A const.)
 $y' = Ae^t$

$$y' + 2y = e^t \Rightarrow Ae^t + 2(Ae^t) = e^t$$

$$A = 1/3$$

$$y_p = \underline{\underline{\frac{1}{3}e^t}}$$

$$3Ae^t = e^t$$

$$3A = 1 \Rightarrow A = 1/3$$

4. a) $3t^2 - 2t + 2y \cdot y' = 0$

Exact? $p = 3t^2 - 2t = h'_t$
 $q = 2y = h'_y$

I: $h = \int (3t^2 - 2t) dt = t^3 - t^2 + g(y)$

II: $h'_y = 0 + g'(y) = 2y \Rightarrow g(y) = y^2 + C = y^2 \leftarrow \begin{matrix} \text{may} \\ \text{choose} \\ C=0 \end{matrix}$

II
 $h = t^3 - t^2 + y^2$ satisfies I-II, eqn. is exact

Solution: $t^3 - t^2 + y^2 = C \leftarrow h(t,y) = C$
 $y^2 = C + t^2 - t^3$
 $y = \pm \sqrt{C + t^2 - t^3}$

b) $2y - 3t^2 + 2(y+t)y' = 0$

$p = 2y - 3t^2 = h'_t \Rightarrow h = 2yt - t^3 + g(y)$
 $q = 2y + 2t = h'_y \Rightarrow h'_y = 2t - 0 + g'(y) = 2y + 2t$
 $g'(y) = 2y$
 $g(y) = y^2 + C = y^2$

$h = 2yt - t^3 + y^2$ exact

$2yt - t^3 + y^2 = C$

$y^2 + 2t \cdot y + (-t^3 - C) = 0$

$y = \frac{-2t \pm \sqrt{4t^2 - 4 \cdot (-t^3 - C)}}{2}$

$= -t \pm \sqrt{t^2 + t^3 + C}$

$$c) \frac{y(1-2\ln t)}{t^3} + \frac{\ln t}{t^2} y' = 0$$

$$p = \frac{y(1-2\ln t)}{t^3} = h'_t$$

$$q = \frac{\ln t}{t^2} = h'_y$$

Start with second eqn:

$$\Rightarrow h = \frac{\ln t}{t^2} \cdot y + g(t)$$

$$h'_t = \frac{\frac{1}{t} \cdot t^2 - \ln t \cdot 2t}{t^4} y + g'(t)$$

$$= \frac{t - 2t \ln t}{t^4} y + g'(t)$$

$$\leftarrow \begin{array}{l} \text{should} \\ \text{equal} \\ p \end{array} = \frac{1 - 2\ln t}{t^3} y + g'(t)$$

ok if $g'(t) = 0$
choose $g(t) = 0$

$$h = \frac{\ln t}{t^2} \cdot y = c$$

$$y = \underline{\underline{\frac{ct^2}{\ln t}}}$$